Curry

An Integrated Functional Logic Language

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Contents

1 Introduction **3**

2 Programs **3**
- 2.1 Datatype Declarations . . . 3
- 2.2 Type Synonym Declarations . . . 4
- 2.3 Function Declarations . . . 5
  - 2.3.1 Functions vs. Variables . . . 6
  - 2.3.2 Conditional Equations with Boolean Guards . . . 7
- 2.4 Local Declarations . . . 7
- 2.5 Free Variables . . . 9
- 2.6 Constraints and Equality . . . 10
- 2.7 Higher-order Features . . . 11

3 Operational Semantics **12**

4 Types **14**
- 4.1 Built-in Types . . . 14
  - 4.1.1 Boolean Values . . . 14
  - 4.1.2 Constraints . . . 15
  - 4.1.3 Functions . . . 16
  - 4.1.4 Integers . . . 16
  - 4.1.5 Floating Point Numbers . . . 17
  - 4.1.6 Lists . . . 17
  - 4.1.7 Characters . . . 17
  - 4.1.8 Strings . . . 18
  - 4.1.9 Tuples . . . 18
- 4.2 Type System . . . 18

5 Expressions **20**
- 5.1 Arithmetic Sequences . . . 20
- 5.2 List Comprehensions . . . 21
- 5.3 Case Expressions . . . 22

6 Modules **23**

7 Input/Output **27**
- 7.1 Monadic I/O . . . 27
- 7.2 Do Notation . . . 28

8 Encapsulated Search **29**
- 8.1 Search Goals and Search Operators . . . 29
- 8.2 Local Variables . . . 31
- 8.3 Predefined Search Operators . . . 32
- 8.4 Choice . . . 33

9 Interface to External Functions and Constraint Solvers **34**
9.1 External Functions . . . 35
9.2 External Constraint Solvers . . . 35

10 Literate Programming **36**

11 Interactive Programming Environment **36**

A Example Programs **38**
A.1 Operations on Lists . . . 38
A.2 Higher-Order Functions . . . 39
A.3 Relational Programming . . . 40
A.4 Functional Logic Programming . . . 41
A.5 Constraint Solving and Concurrent Programming . . . 42
A.6 Concurrent Object-Oriented Programming . . . 44

B Standard Prelude **46**

C Syntax of Curry **59**
C.1 Lexicon . . . 59
C.2 Layout . . . 60
C.3 Context Free Syntax . . . 60
C.4 Infix Operators . . . 63

D Operational Semantics of Curry **64**
D.1 Definitional Trees . . . 64
D.2 Computation Steps . . . 66
D.3 Committed Choice . . . 68
D.4 Equational Constraints . . . 68
D.5 Higher-Order Features . . . 69
D.6 Generating Definitional Trees . . . 70
D.7 Encapsulated Search . . . 72
D.8 Eliminating Local Declarations . . . 74

Bibliography **78**

Index **82**

1 Introduction

Curry is a universal programming language aiming at the amalgamation of the most important declarative programming paradigms, namely functional programming and logic programming. Curry combines in a seamless way features from functional programming (nested expressions, lazy evaluation, higher-order functions), logic programming (logical variables, partial data structures, built-in search), and concurrent programming (concurrent evaluation of constraints with synchronization on logical variables). Moreover, Curry provides additional features in comparison to the pure languages (compared to functional programming: search, computing with partial information; compared to logic programming: more efficient evaluation due to the deterministic evaluation of functions). Moreover, it also amalgamates the most important operational principles developed in the area of integrated functional logic languages: "residuation" and "narrowing" (see [21] for a survey on functional logic programming).

The development of Curry is an international initiative intended to provide a common platform for the research, teaching[1] and application of integrated functional logic languages. This document describes the features of Curry, its syntax and operational semantics.

2 Programs

A Curry program specifies the semantics of expressions, where goals, which occur in logic programming, are particular expressions (of type `Success`, cf. Section 4.1.2, page 15). Executing a Curry program means simplifying an expression until a value (or solution) is computed. To distinguish between values and reducible expressions, Curry has a strict distinction between (data) *constructors* and *operations* or *defined functions* on these data. Hence, a Curry program consists of a set of type and function declarations. The type declarations define the computational domains (constructors) and the function declarations the operations on these domains. Predicates in the logic programming sense can be considered as constraints, i.e., functions with result type `Success`.

Modern functional languages (e.g., Haskell [27], SML [36]) allow the detection of many programming errors at compile time by the use of polymorphic type systems. Similar type systems are also used in modern logic languages (e.g., Gödel [26], λProlog [38]). Curry follows the same approach, i.e., it is a strongly typed language with a Hindley/Milner-like polymorphic type system [13].[2] Each object in a program has a unique type, where the types of variables and operations can be omitted and are reconstructed by a type inference mechanism.

2.1 Datatype Declarations

A datatype declaration has the form

$$\texttt{data}\ T\ \alpha_1\ \ldots\ \alpha_n\ \texttt{=}\ C_1\ \tau_{11}\ \ldots\ \tau_{1n_1}\ \texttt{|}\ \cdots\ \texttt{|}\ C_k\ \tau_{k1}\ \ldots\ \tau_{kn_k}$$

[1]Actually, Curry has been successfully applied to teach functional and logic programming techniques in a single course without switching between different programming languages. More details about this aspect can be found in [22].

[2]The extension of this type system to Haskell's type classes is not included in the kernel language but could be considered in a future version.

and introduces a new n-ary *type constructor* T and k new (data) *constructors* $C_1, \ldots, C_k$, where each C_i has the type

τ_{i1} `->` $\cdots$ `->` τ_{in_i} `->` $T\ \alpha_1 \ldots \alpha_n$

($i = 1, \ldots, k$). Each τ_{ij} is a *type expression* built from the *type variables* $\alpha_1, \ldots, \alpha_n$ and some type constructors. Curry has a number of built-in type constructors, like `Bool`, `Int`, `->` (function space), or, lists and tuples, which are described in Section 4.1. Since Curry is a higher-order language, the types of functions (i.e., constructors or operations) are written in their curried form τ_1 `->` τ_2 `->` $\cdots$ `->` τ_n `->` τ where τ is not a functional type. In this case, n is called the *arity* of the function. For instance, the datatype declarations

```
data Bool = True | False
data List a = [] | a : List a
data Tree a = Leaf a | Node (Tree a) a (Tree a)
```

introduces the datatype `Bool` with the 0-ary constructors (*constants*) `True` and `False`, and the polymorphic types `List a` and `Tree a` of lists and binary trees. Here, "`:`" is an infix operator, i.e., "`a:List a`" is another notation for "`(:)  a (List a)`". Lists are predefined in Curry, where the notation "`[a]`" is used to denote list types (instead of "`List a`"). The usual convenient notations for lists are supported, i.e., `[0,1,2]` is an abbreviation for `0:(1:(2:[]))` (see also Section 4.1).

A *data term* is a variable x or a constructor application $c\, t_1 \ldots t_n$ where c is an n-ary constructor and $t_1, \ldots, t_n$ are data terms. An *expression* is a variable or a (partial) application $\varphi\ e_1 \ldots e_m$ where φ is a function or constructor and $e_1, \ldots, e_m$ are expressions. A data term or expression is called *ground* if it does not contain any variable. Ground data terms correspond to values in the intended domain, and expressions containing operations should be evaluated to data terms. Note that traditional functional languages compute on ground expressions, whereas logic languages also allow non-ground expressions.

2.2 Type Synonym Declarations

To make type definitions more readable, it is possible to specify new names for type expressions by a *type synonym declaration*. Such a declaration has the general form

`type` $T\ \alpha_1$ `...` α_n `=` τ

which introduces a new n-ary type constructor T. $\alpha_1, \ldots, \alpha_n$ are pairwise distinct type variables and τ is a type expressions built from type constructors and the type variables $\alpha_1, \ldots, \alpha_n$. The type $(T\ \ \tau_1 \ldots \tau_n)$ is equivalent to the type $\{\alpha_1 \mapsto \tau_1, \ldots, \alpha_n \mapsto \tau_n\}(\tau)$, i.e., the type expression τ where each α_i is replaced by τ_i. Thus, a type synonym and its definition are always interchangeable and have no influence on the typing of a program. For example, we can provide an alternative notation for list types and strings by the following type synonym declarations:

```
type List a = [a]
type Name = [Char]
```

Since a type synonym introduces just another name for a type expression, recursive or mutually dependent type synonym declarations are not allowed. Therefore, the following declarations are *invalid*:

```
type RecF a = a -> RecF a   -- recursive definitions not allowed
type Place = [Addr]         -- mutually recursive definitions not allowed
type Addr = [Place]
```

However, recursive definitions with an intervening datatype are allowed, since recursive datatype definitions are also allowed. For instance, the following definitions are valid:

```
type Place = [Addr]
data Addr = Tag [Place]
```

2.3 Function Declarations

A function is defined by a type declaration (which can be omitted) followed by a list of defining equations. A *type declaration* for an n-ary function has the form

$$f \texttt{ :: } \tau_1 \texttt{ -> } \tau_2 \texttt{ -> } \cdots \texttt{ -> } \tau_n \texttt{ -> } \tau$$

where $\tau_1, \ldots, \tau_n, \tau$ are type expressions and τ is not a functional type. The simplest form of a *defining equation* (or *rule*) for an n-ary function f ($n \geq 0$) is

$$f\ t_1 \ldots\ t_n \texttt{ = } e$$

where $t_1, \ldots, t_n$ are data terms and the *right-hand side* e is an expression. The *left-hand side* $f\ t_1 \ldots t_n$ must not contain multiple occurrences of a variable. Functions can also be defined by *conditional equations* which have the form

$$f\ t_1 \ldots\ t_n \texttt{ | } c \texttt{ = } e$$

where the *condition* c is a constraint (cf. Section 2.6). In order to apply a conditional equation, its condition must be solved.

Note that one can define functions with a non-determinate behavior by providing several rules with overlapping left-hand sides or *free variables* (i.e., variables which do not occur in the left-hand side) in the conditions or right-hand sides of rules. For instance, the following non-deterministic function inserts an element at an arbitrary position in a list:

```
insert x []     = [x]
insert x (y:ys) = x : y : ys
insert x (y:ys) = y : insert x ys
```

Such *non-deterministic functions* can be given a perfect declarative semantics [18] and their implementation causes no overhead in Curry since techniques for handling non-determinism are already contained in the logic programming part (see also [4]). However, deterministic functions are advantageous since they provide for more efficient implementations (like the *dynamic cut* [34]). If one is interested only in defining deterministic functions, this can be ensured by the following restrictions:

1. Each variable occurring in the right-hand side of a rule must also occur in the corresponding left-hand side.

2. If $f\ t_1 \ldots t_n \texttt{ | } c \texttt{ = } e$ and $f\ t'_1 \ldots t'_n \texttt{ | } c' \texttt{ = } e'$ are rules defining f and σ is a *substitu-*

tion[3] with $\sigma(t_1 \dots t_n) = \sigma(t'_1 \dots t'_n)$, then at least one of the following conditions holds:

(a) $\sigma(e) = \sigma(e')$ (*compatibility of right-hand sides*).

(b) $\sigma(c)$ and $\sigma(c')$ are not simultaneously satisfiable (*incompatibility of conditions*). A decidable approximation of this condition can be found in [30].

These conditions ensure the confluence of the rules if they are considered as a conditional term rewriting system [47]. Implementations of Curry may check these conditions and warn the user if they are not satisfied. There are also more general conditions to ensure confluence [47] which can be checked instead of the above conditions.

Note that defining *equations of higher-type*, e.g., `f = g` if `f` and `g` are of type `Bool -> Bool`, are formally excluded in order to define a simple operational semantics based on first-order rewriting.[4] For convenience, a defining equation `f = g` between functions is allowed but will be interpreted in Curry as syntactic sugar for the corresponding defining equation `f x = g x` on base types.

2.3.1 Functions vs. Variables

In lazy functional languages, different occurrences of the same variable are shared to avoid multiple evaluations of identical expressions. For instance, if we apply the rule

```
double x = x+x
```

to an expression `double` t, we obtain the new expression t+t but both occurrences of t denote the identical expression, i.e., these subterms will be simultaneously evaluated. Thus, *several occurrences of the same variable are always shared*, i.e., if one occurrence of an argument variable, which might bound to an evaluable expression when the function is applied, is evaluated to some value, all other occurrences of this variable are replaced by the same value (without evaluating these other occurrences again). This sharing is necessary not only for efficiency reasons but it has also an influence on the soundness of the operational semantics in the presence of non-deterministic functions (see also [18]). For instance, consider the non-deterministic function `coin` defined by the rules

```
coin = 0
coin = 1
```

Thus, the expression `coin` evaluates to `0` or `1`. However, the result values of the expression `(double coin)` depend on the sharing of the two occurrences of `coin` after applying the rule for `double`: if both occurrences are shared (as in Curry), the results are `0` or `2`, otherwise (without sharing) the results are `0`, `1`, or `2`. The sharing of argument variables corresponds to the so-called "call time choice" in the declarative semantics [18]: if a rule is applied to a function call, a unique value must be assigned to each argument (in the example above: either `0` or `1` must be assigned to the occurrence of `coin` when the expression `(double coin)` is evaluated). This does not mean

[3] A *substitution* σ is a mapping from variables into expressions which is extended to a homomorphism on expressions by defining $\sigma(f\ t_1 \dots t_n) = f\ \sigma(t_1) \dots \sigma(t_n)$. $\{x_1 \mapsto e_1, \dots, x_k \mapsto e_k\}$ denotes the substitution σ with $\sigma(x_i) = e_i$ $(i = 1, \dots, k)$ and $\sigma(x) = x$ for all variables $x \notin \{x_1, \dots, x_k\}$.

[4] Note that this is also necessary in an extension of Curry which allows higher-order rewrite rules, since rules with lambda-abstractions in left-hand sides which are not of base type may cause a gap between the standard notion of higher-order rewriting and the corresponding equational theory [41].

that functions have to be evaluated in a strict manner but this behavior can be easily obtained by sharing the different occurrences of a variable.

Since different occurrences of the same variable are always shared but different occurrences of (textually identical) function calls are not shared, it is important to distinguish between variables and functions. Usually, all symbols occurring at the top-level in the left-hand side of some rule are considered as functions and the non-constructor symbols in the arguments of the left-hand sides are considered as variables. But note that there is a small exception from this general rule in local declarations (see Section 2.4).

2.3.2 Conditional Equations with Boolean Guards

For convenience and compatibility with Haskell, one can also write conditional equations with multiple guards

$$
\begin{array}{ll}
f\ t_1 \ldots\ t_n & \texttt{|}\ b_1\ \texttt{=}\ e_1 \\
 & \vdots \\
 & \texttt{|}\ b_k\ \texttt{=}\ e_k
\end{array}
$$

where $b_1, \ldots, b_k$ $(k > 0)$ are expressions of type `Bool`. Such a rule is interpreted as in Haskell: the guards are successively evaluated and the right-hand side of the first guard which is evaluated to `True` is the result of applying this equation. Thus, this equation can be considered as an abbreviation for the rule

$$
\begin{array}{ll}
f\ t_1 \ldots\ t_n\ \texttt{=} & \texttt{if}\ b_1\ \texttt{then}\ e_1\ \texttt{else} \\
 & \vdots \\
 & \texttt{if}\ b_k\ \texttt{then}\ e_k\ \texttt{else}\ \textit{undefined}
\end{array}
$$

(where *undefined* is some non-reducible function). To write rules with several Boolean guards more nicely, there is a Boolean function `otherwise` which is predefined as `True`. For instance, the factorial function can be declared as follows:

```
fac n | n==0      = 1
      | otherwise = fac(n-1)*n
```

Since all guards have type Bool, this definition is equivalent to (after a simple optimization)

```
fac n = if n==0 then 1 else fac(n-1)*n
```

To avoid confusion, it is not allowed to mix the Haskell-like notation with Boolean guards and the standard notation with constraints: in a rule with multiple guards, all guards must be expressions of type `Bool`. The default type of a guard in single-guarded rules is `Success`, i.e., in a rule like

```
f c x | c = x
```

the type of the variable `c` is `Success` (provided that it is not restricted to `Bool` by the use of `f`).

2.4 Local Declarations

Since not all auxiliary functions should be globally visible, it is possible to restrict the scope of declared entities. Note that the scope of parameters in function definitions is already restricted

since the variables occurring in parameters of the left-hand side are only visible in the corresponding conditions and right-hand sides. The visibility of other entities can be restricted using `let` in expressions or `where` in defining equations.

An expression of the form `let` *decls* `in` *exp* introduces a set of local names. The list of local declarations *decls* can contain function definitions as well as definitions of constants by pattern matching. The names introduced in these declarations are visible in the expression *exp* and the right-hand sides of the declarations in *decls*, i.e., the local declarations can be mutually recursive. For instance, the expression

```
let a=3*b
    b=6
in 4*a
```

reduces to the value `72`.

Auxiliary functions which are only introduced to define another function should often not be visible outside. Therefore, such functions can be declared in a `where`-clause added to the right-hand side of the corresponding function definition. This is done in the following definition of a fast exponentiation where the auxiliary functions `even` and `square` are only visible in the right-hand side of the rule for `exp`:

```
exp b n = if n==0
          then 1
          else if even n then square (exp b (n `div` 2))
                         else b * (exp b (n-1))
       where even n = n `mod` 2 == 0
             square n = n*n
```

Similarly to `let`, `where`-clauses can contain mutually recursive function definitions as well as definitions of constants by pattern matching. The names declared in the `where`-clauses are only visible in the corresponding conditions and right-hand sides. As a further example, the following Curry program implements the quicksort algorithm with a function `split` which splits a list into two lists containing the smaller and larger elements:

```
split e []              = ([],[])
split e (x:xs) | e>=x = (x:l,r)
               | e<x  = (l,x:r)
               where
                 (l,r) = split e xs

qsort []     = []
qsort (x:xs) = qsort l ++ (x:qsort r)
             where
               (l,r) = split x xs
```

To distinguish between locally introduced functions and variables (see also Section 2.3.1), we define a *local pattern* as a (variable) identifier or an application where the top symbol is a data constructor. If the left-hand side of a local declaration in a `let` or `where` is a pattern, then all identifiers in

this pattern that are not data constructors are considered as variables. For instance, the locally introduced identifiers `a`, `b`, `l`, and `r` in the previous examples are variables whereas the identifiers `even` and `square` denote functions. Note that this rule exclude the definition of 0-ary local functions since a definition of the form "`where f = ...`" is considered as the definition of a local variable `f` by this rule which is usually the intended interpretation (see previous examples). Appendix D.8 contains a precise formalization of the meaning of local definitions.

2.5 Free Variables

Since Curry is intended to cover functional as well as logic programming paradigms, expressions (or constraints, see Section 2.6) might contain free (unbound, uninstantiated) variables. The idea is to compute values for these variables such that the expression is reducible to a data term or that the constraint is solvable. For instance, consider the definitions

```
mother John    = Christine
mother Alice   = Christine
mother Andrew  = Alice
```

Then we can compute a child of `Alice` by solving the equation (see Section 2.6) `mother x =:= Alice`. Here, `x` is a free variable which is instantiated to `Andrew` in order to reduce the equation's left-hand side to `Alice`. Similarly, we can compute a grandchild of `Chistine` by solving the equation `mother (mother x) =:= Christine` which yields the value `Andrew` for `x`.

In logic programming languages like Prolog, all free variables are considered as existentially quantified at the top-level. Thus, they are always implicitly declared at the top-level. In a language with different nested scopes like Curry, it is not clear to which scope an undeclared variable belongs (the exact scope of a variable becomes particularly important in the presence of search operators, see Section 8, where existential quantifiers and lambda abstractions are often mixed). Therefore, Curry requires that *each free variable x must be explicitly declared* using a declaration of the form x `free`. These declarations can occur in `where`-clauses or in a `let` enclosing a constraint. The variable is then introduced as unbound with the same scoping rules as for all other local entities (see Section 2.4). For instance, we can define

```
isgrandmother g | let c free in mother (mother c) =:= g  = True
```

As a further example, consider the definition of the concatentation of two lists:

```
append []     ys = ys
append (x:xs) ys = x : append xs ys
```

Then we can define the function `last` which computes the last element of a list by the rule

```
last l | append xs [x] =:= l  = x  where x,xs free
```

Since the variable `xs` occurs in the condition but not in the right-hand side, the following definition is also possible:

```
last l | let xs free in append xs [x] =:= l  = x  where x free
```

Note that the `free` declarations can be freely mixed with other local declarations after a `let` or `where`. The only difference is that a declaration like "`let x free`" introduces an existentially quan-

tified variable (and, thus, it can only occur in front of a constraint) whereas other `let` declarations introduce local functions or parameters. Since all local declarations can be mutually recursive, it is also possible to use local variables in the bodies of the local functions in one `let` declarations. For instance, the following expression is valid (where the functions `h` and `k` are defined elsewhere):

```
let f x = h x y
    y free
    g z = k y z
in c y (f (g 1))
```

Similarly to the usual interpretation of local definitions by lambda lifting [29], this expression can be interpreted by transforming the local definitions for `f` and `g` into global ones by adding the non-local variables of the bodies as parameters:

```
f y x = h x y
g y z = k y z
...
  let y free in c y (f y (g y 1))
```

See Appendix D.8 for more details about the meaning and transformation of local definitions.

2.6 Constraints and Equality

A condition of a rule is a constraint which must be solved in order to apply the rule. An elementary constraint is an *equational constraint* e_1`=:=`e_2 between two expressions (of base type). e_1`=:=`e_2 is satisfied if both sides are reducible to a same ground data term. This notion of equality, which is the only sensible notion of equality in the presence of non-terminating functions [17, 37] and also used in functional languages, is also called *strict equality*. As a consequence, if one side is undefined (nonterminating), then the strict equality does not hold. Operationally, an equational constraint e_1`=:=`e_2 is solved by evaluating e_1 and e_2 to unifiable data terms. The equational constraint could also be solved in an incremental way by an interleaved lazy evaluation of the expressions and binding of variables to constructor terms (see [33] or Section D.4 in the appendix).

Equational constraints should be distinguished from standard Boolean functions (cf. Section 4.1) since constraints are checked for satisfiability. For instance, the equational constraint `[x]=:=[0]` is satisfiable if the variable `x` is bound to `0`. However, the evaluation of `[x]=:=[0]` does not deliver a Boolean value `True` or `False`, since the latter value would require a binding of `x` to all values different from `0` (which could be expressed if a richer constraint system than substitutions, e.g., disequality constraints [7], is used). This is sufficient since, similarly to logic programming, constraints are only activated in conditions of equations which must be checked for satisfiability.

Note that the basic kernel of Curry only provides strict equations e_1`=:=`e_2 between expressions as elementary constraints. Since it is conceptually fairly easy to add other constraint structures [35], extensions of Curry may provide richer constraint systems to support constraint logic programming applications.

Constraints can be combined into a *conjunction* written as c_1 `&` c_2. The conjunction is interpreted *concurrently*: if the combined constraint c_1 `&` c_2 should be solved, c_1 and c_2 are solved concurrently. In particular, if the evaluation of c_1 suspends, the evaluation of c_2 can proceed which

may cause the reactivation of c_1 at some later time (due to the binding of common variables). In a sequential implementation, the evaluation of $c_1\,\&\,c_2$ could be started by an attempt to solve c_1. If the evaluation of c_1 suspends, an evaluation step is applied to c_2.

It is interesting to note that parallel functional computation models [11, 12] are covered by the use of concurrent constraints. For instance, a constraint of the form

```
x =:= f t1  &  y =:= g t2  &  z =:= h x y
```

specifies a potentially concurrent computation of the functions `f`, `g` and `h` where the function `h` can proceed its computation only if the arguments have been bound by evaluating the expressions `f t1` and `g t2`. Since constraints could be passed as arguments or results of functions (like any other data object or function), it is possible to specify general operators to create flexible communication architectures similarly to Goffin [12]. Thus, the same abstraction facilities could be used for sequential as well as concurrent programming. On the other hand, the clear separation between sequential and concurrent computations (e.g., a program without any occurrences of concurrent conjunctions is purely sequential) supports the use of efficient and optimal evaluation strategies for the sequential parts [4, 6], where similar techniques for the concurrent parts are not available.

2.7 Higher-order Features

Curry is a higher-order language supporting the common functional programming techniques by partial function applications and lambda abstractions. *Function application* is denoted by juxtaposition the function and its argument. For instance, the well-known `map` function is defined in Curry by

```
map :: (a -> b) -> [a] -> [b]
map f []     = []
map f (x:xs) = f x : map f xs
```

However, there is an important difference w.r.t. to functional programming. Since Curry is also a logic language, it allows logical variables also for functional values, i.e., it is possible to evaluate the equation `map f [1 2] =:= [2 3]` which has, for instance, a solution {`f=inc`} if `inc` is the increment function on natural numbers. In principle, such solutions can be computed by extending (first-order) unification to *higher-order unification* [24, 38, 43]. Since higher-order unification is a computationally expensive operation, Curry delays the application of unknown functions until the function becomes known [2, 46].[5] Thus, the application operation can be considered as a function ("`@`" is a left-associative infix operator)

```
(@) :: (a -> b) -> a -> b
f@x = f x
```

which is "rigid" in its first argument (cf. Section 3).

In cases where a function is only used a single time, it is tedious to assign a name to it. For

[5]Note that an unbound functional variable can never be instantiated if all program rules are constructor-based and the equational constraint `=:=` denotes equality between data terms. However, extensions of Curry might overcome this weakness by instantiating unbound functional variables to (type-conform) functions occurring in the program in order to evaluate an application (as in [19]), or by considering partial applications (i.e., functions calls with less than the required number of arguments) as data terms (as in [49]).

such cases, anonymous functions (λ-*abstractions*), denoted by

$\backslash x_1 \ldots x_n$->$e$

are provided.

3 Operational Semantics

Curry's operational semantics is based on the lazy evaluation of expressions combined with a possible instantiation of free variables occurring in the expression. If the expression is ground, the operational model is similar to lazy functional languages, otherwise (possibly non-deterministic) variable instantiations are performed as in logic programming. If an expression contains free variables, it may be reduced to different values by binding the free variables to different expressions. In functional programming, one is interested in the computed *value*, whereas logic programming emphasizes the different bindings (*answers*). Thus, we define for the integrated functional logic language Curry an *answer expression* as a pair "$\sigma \,[\!]\, e$" consisting of a substitution σ (the answer computed so far) and an expression e. An answer expression $\sigma \,[\!]\, e$ is *solved* if e is a data term. Usually, the identity substitution in answer expressions is omitted, i.e., we write e instead of $\{\} \,[\!]\, e$ if it is clear from the context.

Since more than one answer may exist for expressions containing free variables, in general, initial expressions are reduced to disjunctions of answer expressions. Thus, a *disjunctive expression* is a (multi-)set of answer expressions $\{\sigma_1 \,[\!]\, e_1, \ldots, \sigma_n \,[\!]\, e_n\}$. For the sake of readability, we write concrete disjunctive expressions like

$$\{\{\mathtt{x} = \mathtt{0}, \mathtt{y} = \mathtt{2}\} \,[\!]\, \mathtt{2}\ ,\ \{\mathtt{x} = \mathtt{1}, \mathtt{y} = \mathtt{2}\} \,[\!]\, \mathtt{3}\}$$

in the form {x=0,y=2} 2 | {x=1,y=2} 3. Thus, substitutions are represented by lists of equations enclosed in curly brackets, and disjunctions are separated by vertical bars.

A single *computation step* performs a reduction in exactly one unsolved expression of a disjunction (e.g., in the leftmost unsolved answer expression in Prolog-like implementations). If the computation step is deterministic, the expression is reduced to a new one. If the computation step is non-deterministic, the expression is replaced by a disjunction of new expressions. The precise behavior depends on the function calls occurring in the expression. For instance, consider the following rules:

```
f 0 = 2
f 1 = 3
```

The result of evaluating the expression `f 1` is `3`, whereas the expression `f x` should be evaluated to the disjunctive expression

```
{x=0} 2 | {x=1} 3 .
```

To avoid superfluous computation steps and to apply programming techniques of modern functional languages, nested expressions are evaluated lazily, i.e., the leftmost outermost function call is primarily selected in a computation step. Due to the presence of free variables in expressions, this function call may have a free variable at an argument position where a value is demanded by the left-hand sides of the function's rules (a value is *demanded* by an argument position of the left-hand

side of some rule, if the left-hand side has a constructor at this position, i.e., in order to apply the rule, the actual value at this position must be the constructor). In this situation there are two possibilities to proceed:

1. Delay the evaluation of this function call until the corresponding free variable is bound (this corresponds to the *residuation* principle which is the basis of languages like Escher [31, 32], Le Fun [2], Life [1], NUE-Prolog [40], or Oz [46]). In this case, the function is called *rigid*.

2. (Non-deterministically) instantiate the free variable to the different values demanded by the left-hand sides and apply reduction steps using the different rules (this corresponds to *narrowing* principle which is the basis of languages like ALF [20], Babel [37], K-Leaf [17], LPG [8], or SLOG [16]). In this case, the function is called *flexible*.

Since Curry is an attempt to provide a common platform for different declarative programming styles and the decision about the "right" strategy depends on the definition and intended meaning of the functions, Curry supports both strategies. The precise strategy is specified by *evaluation annotations* for each function.[6] The precise operational meaning of evaluation annotations is defined in Appendix D. A function can be explicitly annotated as *rigid*. If an explicit annotation is not provided by the user, a default strategy is used: functions with the result type "`IO ...`" are rigid and all other defined functions are flexible. Functions with a polymorphic result type (like the identity) are considered as flexible, although they can be applied like a function with result type "`IO ...`" in a particular context. For each external function, i.e., a function that is not defined by explicit program rules, the strategy is also fixed. Most predefined functions, like arithmetic operators (see Section 4.1.4), are rigid, i.e., they suspend if some argument is an unbound variable.

For instance, consider the function `f` as defined above. As the default strategy, `f` is flexible, i.e., the expression `f x` is evaluated by instantiating `x` to `0` or `1` and applying a reduction step in both cases. This yields the disjunctive expression

```
{x=0} 2 | {x=1} 3 .
```

However, if `f` has the evaluation annotation

```
f eval rigid
```

(i.e., `f` is rigid), then the expression `f x` cannot be evaluated since the argument is a free variable. In order to proceed, we need a "generator" for values for `x`, which is the case in the following constraint:

```
f x =:= y  &  x =:= 1
```

Here, the first constraint `f x =:= y` cannot be evaluated and thus suspends, but the second constraint `x=:=1` is evaluated by binding `x` to `1`. After this binding, the first constraint can be evaluated and the entire constraint is solved. Thus, the constraint is solved by the following steps:

```
f x =:= y  &  x =:= 1
⇝ {x=1} f 1 =:= y
⇝ {x=1} 3 =:= y
```

[6]Evaluation annotations are similar to coroutining declarations [39] in Prolog where the programmer specifies conditions under which a literal is ready for a resolution step.

$\rightsquigarrow$ {x=1,y=3}

(The empty constraint is omitted in the final answer.)

4 Types

4.1 Built-in Types

This section describes the types that are predefined in Curry. For each type, some important operations are discussed. The complete definition of all operations can be found in the standard prelude (see Appendix B).

4.1.1 Boolean Values

Boolean values are predefined by the datatype declaration

```
data Bool = True | False
```

The (sequential) conjunction is predefined as the left-associative infix operator `&&`:

```
(&&) :: Bool -> Bool -> Bool
True  && x = x
False && x = False
```

Similarly, the (sequential) disjunction `||` and the negation `not` are defined as usual (see Appendix B). Furthermore, the function `otherwise` is predefined as `True` to write rules with multiple guards more nicely.

Boolean values are mainly used in conditionals, i.e., the conditional function `if_then_else` is predefined as

```
if_then_else :: Bool -> a -> a -> a
if True  then x else y = x
if False then x else y = y
```

where "`if b then x else y`" is syntactic sugar for the application `(if_then_else b x y)`.

A function with result type `Bool` is often called a *predicate* (in contrast to constraints which have result type `Success`, see below).[7] There are a number of built-in predicates for comparing objects, like the predicate "`<`" to compare numbers (e.g., `1<2` evaluates to `True` and `4<3` evaluates to `False`). There is also a standard predicate "`==`" to test the convertibility of expressions to identical data terms ("strict equality", cf. Section 2.6): e_1`==`e_2 reduces to `True` if e_1 and e_2 are reducible to identical ground data terms, and it reduces to `False` if e_1 and e_2 are reducible to different ground data terms. The evaluation of e_1`==`e_2 suspends if one of the arguments is a free variable (i.e., `==` is a "rigid" function, cf. Section 3). If neither e_1 nor e_2 is a free variable, e_1`==`e_2 is reduced according to the following rules:

```
C           == C           = True                       % for all 0-ary constructors C
C x1 ... xn == C y1 ... yn = x1==y1 &&...&& xn==yn      % for all n-ary constructors C
```

[7] Predicates in the logic programming sense should be considered as constraints since they are only checked for satisfiability and usually not reduced to `True` or `False` in contrast to Boolean functions.

```
C x1 ... xn == D y1 ... ym = False        % for all different constructors C and D
```

This implements a test for *strict equality* (cf. Section 2.6). For instance, the test "`[0]==[0]`" reduces to `True`, whereas the test "`1==0`" reduces to `False`. An equality test with a free variable in one side is delayed in order to avoid an infinite set of solutions for insufficiently instantiated tests like `x==y`. Usually, strict equality is only defined on data terms, i.e., `==` is not really polymorphic but an overloaded function symbol. This could be more precisely expressed using type classes which will be considered in a future version.

Note that e_1`==`e_2 only *tests* the identity of e_1 and e_2 but never binds one side to the other if it is a free variable. This is in contrast to solving the equational constraint e_1`=:=`e_2 which is checked for *satisfiability* and propagates new variable bindings in order to solve this constraint. Thus, in terms of concurrent constraint programming languages [44], `==` and `=:=` correspond to *ask* and *tell* equality constraints, respectively.

4.1.2 Constraints

The type `Success` is the result type of expressions used in conditions of defining rules. Since conditions must be successfully evaluated in order to apply the rule, the type `Success` can be also interpreted as the type of successful evaluations. A function with result type `Success` is also called a *constraint*.

Constraints are different from Boolean-valued functions: a Boolean expression reduces to `True` or `False` whereas a constraint is checked for satisfiability. A constraint is applied (i.e., solved) in a condition of a conditional equation. The *equational constraint* e_1`=:=`e_2 is an elementary constraint which is solvable if the expressions e_1 and e_2 are evaluable to unifiable data terms. `success` denotes an always solvable constraint.

Constraints can be combined into a conjunction of the form c_1 `&` c_2 `&` ... `&` c_n by applying the concurrent conjunction operator `&`. In this case, all constraints in the conjunction are evaluated concurrently. Constraints can also be evaluated in a sequential order by the sequential conjunction operator `&>`, i.e., the combined constraint c_1 `&>` c_2 will be evaluated by first completely evaluating c_1 and then c_2.

Constraints can be passed as parameters or results of functions like any other data object. For instance, the following function takes a list of constraints as input and produces a single constraint, the concurrent conjunction of all constraints of the input list:

```
conj :: [Success] -> Success
conj []     = success
conj (c:cs) = c & conj cs
```

The trivial constraint `success` is usually not shown in answers to a constraint expression. For instance, the constraint `x*x=:=y & x=:=2` is evaluated to the answer

```
{x=2, y=4}
```

4.1.3 Functions

The type `t1 -> t2` is the type of a function which produces a value of type `t2` for each argument of type `t1`. A function f is applied to an argument x by writing "$f\ x$". The type expression

t_1 `->` t_2 `->` $\cdots$ `->` t_{n+1}

is an abbreviation for the type

t_1 `-> (`t_2 `-> (`$\cdots$ `->` t_{n+1}`))`

and denotes the type of a (currified) n-ary function, i.e., `->` associates to the right. Similarly, the expression

$f\ e_1\ e_2\ \ldots\ e_n$

is an abbreviation for the expression

`(...((`$f\ e_1$`)` e_2`)` $\ldots\ e_n$`)`

and denotes the application of a function f to n arguments, i.e., the application associates to the left.

The prelude also defines a right-associative application operator "`$`" which is sometimes useful to avoid brackets. Since `$` has low, right-associative binding precedence, the expression "`f $ g $ 3+4`" is equivalent to "`f (g (3+4))`".

Furthermore, the prelude also defines a right-associative application operator "`$!`" that enforces the evaluation of the argument and suspends if the argument is a free variable. It is based on a predefined (infix) operator `seq` that evaluates the first argument, suspends if this argument is a free variable, and returns the value of the second argument:

```
($!)    :: (a -> b) -> a -> b
f $! x  = x `seq` f x
```

For instance, if `inf` is a non-terminating function (e.g., defined by "`inf = inf`") and `f` a constant function defined by "`f _ = 0`", then the evaluation of the expression "`f $! inf`" does not terminate whereas the expression "`f $ inf`" evaluates to `0`.

4.1.4 Integers

The common integer values, like "42" or "-15", are considered as constructors (constants) of type `Int`. The usual operators on integers, like `+` or `*`, are predefined functions with "rigid" arguments, i.e., are evaluated only if both arguments are integer values, otherwise such function calls are delayed. Thus, these functions can be used as "passive constraints" which become active after binding their arguments. For instance, if the constraint `digit` is defined by the equations

```
digit 0 = success
...
digit 9 = success
```

then the constraint `x*x=:=y & x+x=:=y & digit x` is solved by delaying the two equations which will be activated after binding the variable `x` to a digit by the constraint `digit x`. Thus, the

corresponding computed solution is

```
{x=0,y=0} | {x=2,y=4}
```

4.1.5 Floating Point Numbers

Similarly to integers, values like “3.14159” or “5.0e-4” are considered as constructors of type `Float`. Since overloading is not included in the kernel version of Curry, the names of arithmetic functions on floats are different from the corresponding functions on integers.

4.1.6 Lists

The type `[t]` denotes all lists whose elements are values of type `t`. The type of lists can be considered as predefined by the declaration

```
data [a] = [] | a : [a]
```

where `[]` denotes the empty list and `x:xs` is the non-empty list consisting of the first element `x` and the remaining list `xs`. Since it is common to denote lists with square brackets, the following convenient notation is supported:

[e_1,e_2,...,e_n]

denotes the list e_1:e_2:$\cdots$:e_n:[] (which is equivalent to e_1:(e_2:($\cdots$:(e_n:[])...)) since “:” is a right-associative infix operator). Note that there is an overloading in the notation `[t]`: if `t` is a type, `[t]` denotes the type of lists containing elements of type `t`, where `[t]` denotes a single element list (with element `t`) if `t` is an expression. Since there is a strong distinction between occurrences of types and expressions, this overloading can always be resolved.

For instance, the following predefined functions define the concatenation of two lists and the application of a function to all elements in a list:

```
(++) :: [a] -> [a] -> [a]
[]     ++ ys = ys
(x:xs) ++ ys = x : xs ++ ys

map :: (a -> b)  -> [a] -> [b]
map f []     = []
map f (x:xs) = f x : map f xs
```

4.1.7 Characters

Values like `'a'` or `'0'` denote constants of type `Char`. There are two conversion functions between characters and their corresponding ASCII values:

```
ord :: Char -> Int
chr :: Int -> Char
```

4.1.8 Strings

The type `String` is an abbreviation for `[Char]`, i.e., strings are considered as lists of characters. String constants are enclosed in double quotes. Thus, the string constant `"Hello"` is identical to the character list `['H','e','l','l','o']`. A term can be converted into a string by the function

```
show :: a -> String
```

For instance, the result of `(show 42)` is the character list `['4','2']`.

4.1.9 Tuples

If $t_1, t_2, \ldots, t_n$ are types and $n \geq 2$, then `(`t_1`,`t_2`,...,`t_n`)` denotes the type of all n-tuples. The elements of type `(`t_1`,`t_2`,...,`t_n`)` are `(`x_1`,`x_2`,...,`x_n`)` where x_i is an element of type t_i ($i = 1, \ldots, n$). Thus, for each n, the tuple-building operation `(,...,)` (with $n-1$ commas) can be considered as an n-ary constructor introduced by the pseudo-declaration

```
data (a1,a2,...,an) = (,...,) a1 a2...an
```

where `(`x_1`,`x_2`,...,`x_n`)` is equivalent to the constructor application "`(,...,)` x_1 x_2`...`x_n".

The unit type `()` has only a single element `()` and can be considered as defined by

```
data () = ()
```

Thus, the unit type can also be interpreted as the type of 0-tuples.

4.2 Type System

Curry is a strongly typed language with a Hindley/Milner-like polymorphic type system [13].[8] Each variable, constructor and operation has a unique type, where only the types of constructors have to be declared by datatype declarations (see Section 2.1). The types of functions can be declared (see Section 2.3) but can also be omitted. In the latter case they will be reconstructed by a type inference mechanism.

Note that Curry is an explicitly typed language, i.e., each function has a type. The type can only be omitted if the type inferencer is able to reconstruct it and to insert the missing type declaration. In particular, the type inferencer can reconstruct only those types which are visible in the module (cf. Section 6). Each type inferencer of a Curry implementation must be able to insert the types of the parameters and the free variables (cf. Section 2.5) for each rule. The automatic inference of the types of the defined functions might require further restrictions depending on the type inference method. Therefore, the following definition of a well-typed Curry program assumes that the types of all defined functions are given (either by the programmer or by the type inferencer). A Curry implementation must accept a well-typed program if all types are explicitly provided but should also support the inference of function types according to [13].

A *type expression* is either a type variable, a basic type like `Bool`, `Success`, `Int`, `Float`, `Char` (or any other type constructor of arity 0), or a type constructor application of the form $(T\, \tau_1 \ldots \tau_n)$

[8]The extension of the type system to Haskell's type classes is not included in the kernel language but may be considered in a future version.

$$\text{Axiom:} \quad \frac{}{\mathcal{A} \vdash x :: \tau} \quad \text{if } \tau \text{ is a generic instance of } \mathcal{A}(x)$$

$$\text{Application:} \quad \frac{\mathcal{A} \vdash e_1 :: \tau_1 \to \tau_2 \quad \mathcal{A} \vdash e_2 :: \tau_1}{\mathcal{A} \vdash e_1\, e_2 :: \tau_2}$$

$$\text{Abstraction:} \quad \frac{\mathcal{A}[x/\tau] \vdash e :: \tau'}{\mathcal{A} \vdash \backslash x\texttt{->}e :: \tau \to \tau'} \quad \text{if } \tau \text{ is a type expression}$$

$$\text{Existential:} \quad \frac{\mathcal{A}[x/\tau] \vdash e :: \texttt{Success}}{\mathcal{A} \vdash \texttt{let}\, x\, \texttt{free in}\, e :: \texttt{Success}} \quad \text{if } \tau \text{ is a type expression}$$

$$\text{Conditional:} \quad \frac{\mathcal{A} \vdash e_1 :: \texttt{Bool} \quad \mathcal{A} \vdash e_2 :: \tau \quad \mathcal{A} \vdash e_3 :: \tau}{\mathcal{A} \vdash \texttt{if}\, e_1\, \texttt{then}\, e_2\, \texttt{else}\, e_3 :: \tau}$$

Figure 1: Typing rules for Curry programs

where T is a type constructor of arity n, as defined by a datatype declaration (cf. Section 2.1),[9] and $\tau_1, \ldots, \tau_n$ are type expressions (note that list, tuple and function types have the special syntax `[·]`, `(·,···,·)`, and `·->·` as described above). For instance, `[(Int,a)]->a` is a type expression containing the type variable `a`. A *type scheme* is a type expression with a universal quantification for some type variables, i.e., it has the form $\forall \alpha_1, \ldots, \alpha_n.\tau$ ($n \geq 0$; in case of $n = 0$, the type scheme is equivalent to a type expression). A function type declaration f`::`τ is considered as an assignment of the type scheme $\forall \alpha_1, \ldots, \alpha_n.\tau$ to f, where $\alpha_1, \ldots, \alpha_n$ are all type variables occurring in τ. The type τ is called a *generic instance* of the type scheme $\forall \alpha_1, \ldots, \alpha_n.\tau'$ if there is a substitution $\sigma = \{\alpha_1 \mapsto \tau_1, \ldots, \alpha_n \mapsto \tau_n\}$ on the types with $\sigma(\tau') = \tau$.

The types of all defined functions are collected in a *type environment* $\mathcal{A}$ which is a mapping from identifiers to type schemes. It contains at least the type schemes of the defined functions and an assignment of types for some local variables. An expression e is *well-typed* and has type τ w.r.t. a type environment $\mathcal{A}$ if $\mathcal{A} \vdash e :: \tau$ is derivable according to the inference rules shown in Figure 1.

A defining equation $f\ t_1 \ldots t_n$ `=` e [`where` x `free`] is well-typed w.r.t. a type environment $\mathcal{A}$ if $\mathcal{A}(f) = \forall \alpha_1, \ldots, \alpha_m.\tau$ [and $\mathcal{A}(x)$ is a type expression] and $\mathcal{A} \vdash \backslash t_1\texttt{->} \cdots \backslash t_n\texttt{->}e :: \tau$ is derivable according to the above inference rules. A conditional equation l `|` c `=` r is considered (for the purpose of typing) as syntactic sugar for l `= cond` $c\ r$ where `cond` is a new function with type scheme $\mathcal{A}(\texttt{cond}) = \forall \alpha.\texttt{Success} \to \alpha \to \alpha$.

A *program is well-typed* if all its rules are well-typed with a unique assignment of type schemes to defined functions.[10]

Note that the following recursive definition is a well-typed Curry program according to the definition above (and the type definitions given in the prelude, cf. Appendix B):

```
f :: [a] -> [a]
f x = if length x == 0 then fst (g x x) else x
```

[9]We assume here that all type constructors introduced by type synonyms (cf. Section 2.2) are replaced by their definitions.

[10]Here we assume that all local declarations are eliminated by the transformations described in Appendix D.8.

```
g :: [a] -> [b] -> ([a],[b])
g x y = (f x, f y)

h :: ([Int],[Bool])
h = g [3,4] [True,False]
```

However, if the type declaration for `g` is omitted, the usual type inference algorithms are not able to infer this type.

5 Expressions

Expressions are a fundamental notion of Curry. As introduced in Section 2, functions are defined by equations defining expressions that are equivalent to specific function calls. For instance, the program rule

```
square x = x*x
```

defines that the function call `(square 3)` is equivalent to the expression `(3*3)`.

Expressions occur in conditions and right-hand sides of equations defining functions. A *computation* evaluates an expression to a data term (see Section 3). Expressions are built from constants of a specific data type (e.g., integer constants, character constants, see Section 4), variables, or applications of constructors or functions to expressions. Furthermore, Curry provides some syntactic extensions for expressions that are discussed in this section.

5.1 Arithmetic Sequences

Curry supports two syntactic extensions to define list of elements in a compact way. The first one is a notation for arithmetic sequences. The *arithmetic sequence*

[e_1 , e_2 .. e_3]

denotes a list of integers starting with the first two elements e_1 and e_2 and ending with the element e_3 (where e_2 and e_3 can be omitted). The precise meaning of this notation is defined by the following translation rules:

Arithmetic sequence notation:	**Equivalent to:**
[e ..]	`enumFrom` e
[e_1,e_2 ..]	`enumFromThen` e_1 e_2
[e_1 .. e_2]	`enumFromTo` e_1 e_2
[e_1,e_2 .. e_3]	`enumFromThenTo` e_1 e_2 e_3

The functions for generating the arithmetic sequences, `enumFrom`, `enumFromThen`, `enumFromTo`, and `enumFromThenTo`, are defined in the prelude (see page 53). Thus, the different notations have the following meaning:

- The sequence [e..] denotes the infinite list [e,e+1,e+2,...].

- The sequence `[`e_1`..`e_2`]` denotes the finite list `[`e_1`,`e_1`+1,`e_1`+2,...,`e_2`]`. Note that the list is empty if $e_1 > e_2$.
- The sequence `[`e_1`,`e_2`..]` denotes the infinite list `[`e_1`,`e_1`+`i`,`e_1`+2*`i`,...]` with $i = e_2 - e_1$. Note that i could be positive, negative or zero.
- The sequence `[`e_1`,`e_2`..`e_3`]` denotes the finite list `[`e_1`,`e_1`+`i`,`e_1`+2*`i`,...,`e_3`]` with $i = e_2 - e_1$. Note that e_3 is not contained in this list if there is no integer m with $e_3 = e_1 + m * i$.

For instance, `[0,2..10]` denotes the list `[0,2,4,6,8,10]`.

5.2 List Comprehensions

The second compact notation for lists are *list comprehensions*. They have the general form

`[` e `|` q_1 `, ... ,` q_k `]`

where $k \geq 1$ and each q_i is a qualifier that is either

- a *generator* of the form p`<-`l, where p is a local pattern (i.e., an expression without defined function symbols and without multiple occurrences of the same variable, compare Section 2.4) of type t and l is an expression of type `[`t`]`, or
- a *guard*, i.e., an expression of type `Bool`.

The variables introduced in a local pattern can be used in subsequent qualifiers and the element description e. Such a list comprehension denotes the list of elements which are the result of evaluating e in the environment produced by the nested and depth-first evaluation of the generators satisfying all guards. For instance, the list comprehension

```
[ x | x <- [1..50], x `mod` 7 == 0 ]
```

denotes the list `[7,14,21,28,35,42,49]`, and the list comprehension

```
[ (x,y) | x<-[1,2,3], y<-[4,5] ]
```

denotes the list `[(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)]`.

The precise meaning of list comprehensions is defined by the following translation rules (for the purpose of this definition, we also consider list comprehensions with an empty qualifier list):

List comprehension:	**Equivalent to:**
`[`e `| ]`	`[`e`]`
`[`e `|` b`,` q`]`	`if` b `then [`e`|`q`] else []` (b Boolean guard)

`[`e `|` p`<-`l`,` q`]`	`let ok` p `= [`e`|`q`]`
	`ok` p_1 `= []`
	`...`
	`ok` p_n `= []`
	`in concatMap ok l`
	where $\{p_1, \ldots, p_n\} = complPat(p)$

In the last translation rule, `concatMap` is defined in the prelude (see page 51) and $complPat(p)$ denotes the set of patterns that are incompatible with p. This set is defined by

$$complPat(x) = \{\}$$

for all variables x and

$$\begin{array}{rcl} complPat(C\ p_1 \ldots p_n) & = & \{C_1\ x_{11} \ldots x_{1n_1}, \ldots, C_k\ x_{k1} \ldots x_{kn_k}\} \\ & & \cup \{C\ p_1 \ldots p_{i-1}\ p'\ y_{i+1} \ldots y_n \mid 1 \leq i \leq n,\ p' \in complPat(p_i)\} \end{array}$$

where $C, C_1, \ldots, C_k$ are all the constructors of the result type of C and all x_{ij}, y_i are fresh variables. For instance, $complPat($`[True]`$) = \{$`[]`, `False:zs`, `True:y:ys`$\}$ and $complPat($`(x,y)`$) = \{\}$.

Note that this translation does not work for patterns containing number constants since in this case the complement set of patterns is infinite. Therefore, we modify this scheme as follows. If the pattern p in a list comprehension `[`e `|` p`<-`l`,` q`]` contains the integer, float or character constants $c_1, \ldots, c_k$, we replace them by corresponding fresh variables $z_1, \ldots, z_k$ in the pattern and apply for this modified pattern p' the above translation scheme but with the following modified first rule for `ok`:

```
ok p' = if z1==c1 && ... && zk==ck then [e|q] else []
```

For instance, the list comprehension

```
[x | (2,x) <- [(1,3),(2,4),(3,6)]]
```

(which is evaluated to `[4]`) is translated into the following expression:

```
let ok (y,x) = if y==2 then [x] else []
in  concatMap ok [(1,3),(2,4),(3,6)]
```

5.3 Case Expressions

Case expressions are a convenient notation of sequential pattern matching with defaults. The general form of a case expression is as follows ($e, e_1, \ldots, e_n$ are expressions and the patterns $p_1, \ldots, p_n$ are data terms):

```
case e of
  p1 -> e1
  ...
  pn -> en
```

Note that case expressions use the layout rule (see Section C.2). Thus, the patterns $p_1, \ldots, p_n$ must be vertically aligned. The informal operational meaning of the case expression is as follows. Evaluate e so that it matches a pattern p_i. If this is possible, replace the entire case expression by the corresponding alternative e_i (after replacing the pattern variables occurring in p_i by their actual expressions). If none of the patterns $p_1, \ldots, p_n$ matches, the computation fails. The pattern matching is tried *sequentially*, from top to bottom, and rigid, without binding of free variables occurring in e. Thus, case expressions correspond to rigid functions. Actually, each case expression can be translated into an auxiliary function so that case expressions do not increase the power of the

language (however, implementations of Curry can choose more efficient implementation techniques). For instance, the function

```
swap z = case z of
           [x,y] -> [y,x]
           _     -> z
```

is equivalent to the following definition:

```
swap eval rigid
swap [] = []
swap [x] = [x]
swap [x,y] = [y,x]
swap (x1:x2:x3:xs) = x1:x2:x3:xs
```

Thus, case expressions are a convenient notation for functions with default cases.

6 Modules

A *module* defines a collection of datatypes, constructors and functions which we call *entities* in the following. A module exports some of its entities which can be imported and used by other modules. An entity which is not exported is not accessible from other modules.

A Curry *program* is a collection of modules. There is one main module which is loaded into a Curry system. The modules imported by the main module are implicitly loaded but not visible to the user. After loading the main module, the user can evaluate expressions which contain entities exported by the main module.

There is one distinguished module, named `prelude`, which is implicitly imported into all programs (see also Appendix B). Thus, the entities defined in the prelude (basic functions for arithmetic, list processing etc.) can be always used.

A module always starts with the head which contains at least the name of the module followed by the keyword `where`, like

```
module stack where ...
```

If a program does not contain a module head, the *standard module head* "`module main where`" is implicitly inserted. *Module names* can be given a hierarchical structure by inserting dots which is useful if larger applications should be structured into different subprojects. For instance,

```
company.project1.subproject2.mod4
```

is a valid module name. The dots may reflect the hierarchical file structure where modules are stored. For instance, the module `compiler.symboltable` could be stored in the file `symboltable.curry` in the directory `compiler`. To avoid such long module names when referring to entities of this module by qualification, imported modules can be renamed by the use of "`as`" in an import declaration (see below).

Without any further restrictions in the module head, all entities which are defined by top-level declarations in this module are exported (but not those entities that are imported into this module). In order to modify this standard set of exported entities of a module, an *export list* can be added

to the module head. For instance, a module with the head

```
module stack(stackType, push, pop, newStack) where ...
```

exports the entities `stackType`, `push`, `pop`, and `newStack`. An export list can contain the following entries:

1. *Names of datatypes*: This exports only the datatype, whose name must be accessible in this module, *but not* the constructors of the datatype. The export of a datatype without its constructors allows the definition of *abstract datatypes*.

2. *Datatypes with all constructors*: If the export list contains the entry `t(..)`, then `t` must be a datatype whose name is in the module. In this case, the datatype `t` and all constructors of this datatype, which must be also accessible in this module, are exported.

3. *Names of functions*: This exports the corresponding functions whose names must be accessible in this module. The types occurring in the argument and result type of this function are implicitly exported, otherwise the function may not be applicable outside this module.

4. *Modules*: The set of all accessible entities imported from a module m into the current module (see below) can be exported by a single entry "`module` m" in the export list. For instance, if the head of the module `stack` is defined as above, the module head

   ```
   module queue(module stack, enqueue, dequeue) where ...
   ```

   specifies that the module `queue` exports the entities `stackType`, `push`, `pop`, `newStack`, `enqueue`, and `dequeue`.

The unqualified names of the exported entities of a module must be pairwise different to avoid name clashes in the use of these entities. For instance, the module

```
module m(f,ma.g) where
import ma
f x = ma.g x
```

exports the names `m.f` and `m.g`, i.e., a *qualified entity* consists always of the name of the exported module followed by a dot and the unqualified name of the entity (without spaces between the dot and the names). If module `ma` also exports the entity `f`, then the export declaration

```
module m(f,ma.f) where
import ma
f x = ma.f x
```

is not allowed since the exported name `m.f` cannot be uniquely resolved.

All entities defined by top-level declarations in a module are *always accessible* in this module, i.e., there is no need to qualify the names of top-level declarations. Additionally, the entities exported by another module can be also made accessible in the module by an `import` declaration. An import declaration consists of the name of the imported module and (optionally) a list of entities imported from that module. If the list of imported entities is omitted, all entities exported by that module are imported. For instance, the import declaration

```
import stack
```

imports all entities exported by the module `stack`, whereas the declaration

```
import family(father, grandfather)
```

imports only the entities `father` and `grandfather` from the module `family`, provided that they are exported by `family`. Similarly to export declarations, a datatype name `t` in an import list imports only the datatype without its constructors whereas the form `t(..)` imports the datatype together with all its constructors (provided that they are also exported).

The names of all imported entities are accessible in the current module, i.e., they are equivalent to top-level declarations, provided that their names are not in conflict with other names. For instance, if a function `f` is imported from module `m` but the current module contains a top-level declaration for `f` (which is thus directly accessible in the current module), the imported function is not accessible (without qualification). Similarly, if two identical names are imported from different modules and denote different entities, none of these entities is accessible (without qualification). It is possible to refer to such imported but not directly accessible names by prefixing them with the module identifier (*qualification*). For instance, consider the module `m1` defined by

```
module m1 where
f :: Int -> Int
...
```

and the module `m2` defined by

```
module m2 where
f :: Int -> Int
...
```

together with the main module

```
module main where
import m1
import m2
...
```

Then the names of the imported functions `f` are not directly accessible in the main module but one can refer to the corresponding imported entities by the qualified identifiers `m1.f` or `m2.f`.

Another method to resolve name conflicts between imported entities is the qualification of an imported module. If we change the `main` module to

```
module main where
import qualified m1
import m2
...
```

then the name `f` refers to the entity `m2.f` since all entities imported from `m1` are *only accessible by qualification* like `m1.f`.

A further method to avoid name conflicts is the hiding of imported entities. Consider the following definition:

```
module main where
import m1 hiding (f)
import m2
...
```

The name `f` in the `main` module refers to the entity `m2.f` since the name `f` is not imported from `m1` by the hiding declaration. The hiding clause effects only unqualified names, i.e., the entity `m1.f` is still accessible in the body of the `main` module. Therefore, a hiding clause has no effect in combination with a `qualified` import. Similarly to export declarations, a datatype `t` in a hiding clause hides only the datatype (but not its constructors) whereas the form `t(..)` hides the complete datatype including its constructors.

The effect of several `import` declarations is cumulative, i.e., if an entity is hidden in one import declaration, it can still be imported by another import declaration. For instance, if module `mt` exports a datatype `t` together with its constructors, then the import declarations

```
import mt hiding (t(..))
import mt (t)
```

imports all entities exported by `mt` but only the name `t` of the datatype without its constructors, since the first hiding clause imports everything from `mt` except the complete datatype `t` and the second import specification additionally imports the name of the datatype `t`.

Imported modules can also be given a new local name in the import declaration. For instance, the declaration

```
import m(f) as foo
```

enables access to the name `f` (provided that it is not in conflict with another entity with the same name) and `foo.f` but not to `m.f`. This local renaming enables the abbreviation of long module names and the substitution of different modules without changing the qualifiers inside a module.

Although each name refers to exactly one entity, it is possible that the same entity is referred by different names. For instance, consider the modules defined by

```
module m(f) where
f :: Int -> Int
...
module m1(m.f) where
import m
...
module m2(m.f) where
import m
...
```

together with the main module

```
module main where
import m1
import m2
...
```

Now the names `f`, `m1.f`, and `m2.f` refer to the identical entity, namely the function `f` defined in module `m`. Note that there is no need to qualify `f` in the module `main` since this name is unambiguously resolved to the function `f` defined in module `m`, although it is imported via two different paths.

Qualified names are treated syntactically like unqualified names. In particular, a qualified infix operator like `Complex.+` has the same fixity as the definition of `+` in the module `Complex`, i.e., the expression "`x Complex.+ y`" is syntactically valid. To distinguish between a function composition like "`f . g`", where "`.`" is an infix operator (see Section 14, page 46), and a module qualification, spaces are not allowed in the module qualification while there must be at least one space before the "`.`" if it is used as an infix operator. Thus, "`f . g`" is interpreted as the composition of `f` and `g` whereas "`f.g`" is interpreted as the object `g` imported from module `f`.

The import dependencies between modules must be *non-circular*, i.e., it is not allowed that module m_1 imports module m_2 and module m_2 also imports (directly or indirectly) module m_1.

7 Input/Output

Curry provides a declarative model of I/O by considering I/O operations as transformations on the outside world. In order to avoid dealing with different versions of the outside world, it must be ensured that at each point of a computation only one version of the world is accessible. This is guaranteed by using the monadic I/O approach [42] of Haskell and by requiring that I/O operations are not allowed in program parts where non-deterministic search is possible.

7.1 Monadic I/O

In the monadic I/O approach, the outside "world" is not directly accessible but only through actions which change the world. Thus, the world is encapsulated in an abstract datatype which provides actions to change the world. The type of such actions is `IO t` which is an abbreviation for

```
World -> (t,World)
```

where `World` denotes the type of all states of the outside world. If an action of type `IO t` is applied to a particular world, it yields a value of type `t` and a new (changed) world. For instance, `getChar` is an action which reads a character from the standard input when it is executed, i.e., applied to a world. Therefore, `getChar` has the type `IO Char`. The important point is that values of type `World` are not accessible for the programmer — she/he can only create and compose actions on the world. Thus, a program intended to perform I/O operations has a sequence of actions as the result. These actions are computed and executed when the program is connected to the world by executing it. For instance,

```
getChar :: IO Char
getLine :: IO String
```

are actions which read the next character or the next line from the standard input. The functions

```
putChar  :: Char -> IO ()
putStr   :: String -> IO ()
putStrLn :: String -> IO ()
```

take a character (string) and produces an action which, when applied to a world, puts this character (string) to the standard output (and a line-feed in case of `putStrLn`.

Since an interactive program consists of a sequence of I/O operations, where the order in the sequence is important, there are two operations to compose actions in a particular order. The function

```
(>>) :: IO a -> IO b -> IO b
```

takes two actions as input and yields an action as output. The output action consists of performing the first action followed by the second action, where the produced value of the first action is ignored. If the value of the first action should be taken into account before performing the second action, the actions can be composed by the function

```
(>>=) :: IO a -> (a -> IO b) -> IO b
```

where the second argument is a function taking the value produced by the first action as input and performs another action. For instance, the action

```
getLine >>= putStrLn
```

is of type `IO ()` and copies, when executed, a line from standard input to standard output.

The `return` function

```
return :: a -> IO a
```

is sometimes useful to terminate a sequence of actions and return a result of an I/O operation. Thus, `return v` is an action which does not change the world and returns the value `v`.

To execute an action, it must be the main expression in a program, i.e., interactive programs have type `IO ()`. Since the world cannot be copied (note that the world contains at least the complete file system), non-determinism in relation with I/O operations must be avoided. Thus, the applied action must always be known, i.e., `>>` and `>>=` are rigid in their arguments. Moreover, it is a runtime error if a disjunctive expression (cf. Section 3) $\sigma_1 [\!] e_1$ `|···|` $\sigma_n [\!] e_n$, where the e_i's are of type `IO ()` and $n > 1$, occurs as the top-level expression of a program, since it is unclear in this case which of the disjunctive actions should be applied to the current world. Thus, all possible search must be encapsulated between I/O operations (see Section 8).

Using the evaluation annotations, a compiler is able to detect functions where search is definitely avoided (e.g., if all evaluated positions are declared as `rigid`). Thus, the compiler may warn the user about non-deterministic computations which may occur in I/O actions so that the programmer can encapsulate them.

7.2 Do Notation

To provide a more conventional notation for programming sequences of I/O operations, Curry has a special piece of syntax for writing such sequences. For instance, consider the following expression to read a line form the standard input and to print them together with the string `"Your input: "` on the standard output:

```
getLine >>= \line -> putStr "Your input: " >> putStrLn line
```

Using the do notation, this expression can be written in a more traditional style:

```
do line <- getLine
   putStr "Your input: "
   putStrLn line
```

Note that the do notation is just another syntax for sequences of I/O actions. Thus, `do` expressions can be used wherever an expression of type I/O is required (but note that `do` expressions use the layout rule (see Section C.2) and, therefore, they are usually written in separate program lines). The following table specifies the translation of `do` expressions into the kernel language:

Do notation:	**Equivalent to:**
`do` *expr*	*expr*
`do` *expr* *stmts*	*expr* `>>` `do` *stmts*
`do` *p* `<-` *expr* *stmts*	*expr* `>>=` `\`*p* `->` `do` *stmts*
`do` `let` *decls* *stmts*	`let` *decls* `in` `do` *stmts*

8 Encapsulated Search

Global search, possibly implemented by backtracking, must be avoided in some situations (user-control of efficiency, concurrent computations, non-backtrackable I/O). Hence it is sometimes necessary to encapsulate search, i.e., non-deterministic computations in parts of larger programs. Non-deterministic computations might occur in Curry whenever a function must be evaluated with a free variable at a flexible argument position. In this case, the computation must follow different branches with different bindings applied to the current expression which has the effect that the entire expression is split into (at least) two independent disjunctions. To give the programmer control on the actions taken in this situation, Curry provides a primitive search operator which is the basis to implement sophisticated search strategies. This section sketches the idea of encapsulating search in Curry and describes some predefined search strategies.

8.1 Search Goals and Search Operators

Since search is used to find solutions to some constraint, search is always initiated by a constraint containing a *search variable* for which a solution should be computed.[11] Since the search variable may be bound to different solutions in different disjunctive branches, it must be abstracted. Therefore, a *search goal* has the type `a->Success` where `a` is the type of the values which we are searching for. In particular, if c is a constraint containing a variable x and we are interested in solutions for x, i.e., values for x such that c is satisfied, then the corresponding search goal has the form `\`x`->`c. However, any other expression of the same type can also be used as a search goal.

To control the search performed to find solutions to search goals, Curry has a predefined operator

```
try :: (a->Success) -> [a->Success]
```

[11] The generalization to more than one search variable is straightforward by the use of tuples.

which takes a search goal and produces a list of search goals. The search operator `try` attempts to evaluate the search goal until the computation finishes or performs a non-deterministic splitting. In the latter case, the computation is stopped and the different search goals caused by this splitting are returned as the result list. Thus, an expression of the form `try \`x`->`c can have three possible results:

1. An empty list. This indicates that the search goal `\`x`->`c has no solution. For instance, the expression

   ```
   try \x -> 1=:=2
   ```

   reduces to `[]`.

2. A list containing a single element. In this case, the search goal `\`x`->`c has a single solution represented by the element of this list. For instance, the expression

   ```
   try \x->[x]=:=[0]
   ```

   reduces to `[\x->x=:=0]`. Note that a solution, i.e., a binding for the search variable like the substitution $\{$`x` $\mapsto$ `0`$\}$, can always be presented by an equational constraint like `x=:=0`.

   Generally, a one-element list as a result of `try` has always the form `[\`x`->`x`=:=`e`]` (plus some local variables, see next subsection) where e is fully evaluated, i.e., e does not contain defined functions. Otherwise, this goal might not be solvable due to the definition of equational constraints.

3. A list containing more than one element. In this case, the evaluation of the search goal `\`x`->`c requires a non-deterministic computation step. The different alternatives immediately after this non-deterministic step are represented as elements of this list. For instance, if the function `f` is defined by

   ```
   f a = c
   f b = d
   ```

   then the expression

   ```
   try \x -> f x =:= d
   ```

   reduces to the list `[\x->x=:=a & f a =:= d, \x->x=:=b & f b =:= d]`. This example also shows why the search variable must be abstracted: the alternative bindings cannot be actually performed (since a free variable is only bound to at most one value in each computation thread) but are represented as equational constraints in the search goal.

Note that the search goals of the list in the last example are not further evaluated. This provides the possibility to determine the behavior for non-deterministic computations. For instance, the following function defines a depth-first search operator which collects all solutions of a search goal in a list:

```
solveAll :: (a->Success) -> [a->Success]
solveAll g = collect (try g)
      where collect []         = []
```

```
collect [g]          = [g]
collect (g1:g2:gs) = concat (map solveAll (g1:g2:gs))
```

(`concat` concatenates a list of lists to a single list). For instance, if `append` is the list concatenation, then the expression

```
solveAll \l -> append l [1] =:= [0,1]
```

reduces to `[\l->l=:=[0]]`.

The value computed for the search variable in a search goal can be easily accessed by applying it to a free variable. For instance, the evaluation of the applicative expression

```
(solveAll \l->append l [1] =:= [0,1]) =:= [g] & g x
```

binds the variable `g` to the search goal `[\l->l=:=[0]]` and the variable `x` to the value `[0]` (due to solving the constraint `x=:=[0]`). Based on this idea, there is a predefined function

```
findall :: (a->Success) -> [a]
```

which takes a search goal and collects all solutions (computed by a depth-first search like `solveAll`) for the search variable into a list.

Due to the laziness of the language, search goals with infinitely many solutions cause no problems if one is interested only in finitely many solutions. For instance, a function which computes only the first solution w.r.t. a depth-first search strategy can be defined by

```
first g = head (findall g)
```

Note that `first` is a partial function, i.e., it is undefined if `g` has no solution.

8.2 Local Variables

Some care is necessary if free variables occur in the search goal, like in the goal

```
\l2 -> append l1 l2 =:= [0,1]
```

Here, only the variable `l2` is abstracted in the search goal but `l1` is free. Since non-deterministic bindings cannot be performed during encapsulated search, *free variables are never bound inside encapsulated search*. Thus, if it is necessary to bind a free variable in order to proceed an encapsulated search operation, the computation suspends. For instance, the expression

```
first \l2 -> append l1 l2 =:= [0,1]
```

cannot be evaluated and will be suspended until the variable `l1` is bound by another part of the computation. Thus, the constraint

```
s =:= (first \l2->append l1 l2 =:= [0,1])  &  l1 =:= [0]
```

can be evaluated since the free variable `l1` in the search goal is bound to `[0]`, i.e., this constraint reduces to the answer

```
{l1=[0], s=[1]}
```

In some cases it is reasonable to have unbound variables in search goals, but these variables should be treated as local, i.e., they might have different bindings in different branches of the search. For

instance, if we want to compute the last element of the list `[3,4,5]` based on `append`, we could try to solve the search goal

```
\e -> append l [e] =:= [3,4,5]
```

However, the evaluation of this goal suspends due to the necessary binding of the free variable `l`. This can be avoided by declaring the variable `l` as *local* to the constraint by the use of `let` (see Section 2.5), like in the following expression:

```
first \e -> let l free in append l [e] =:= [3,4,5]
```

Due to the local declaration of the variable `l` (which corresponds logically to an existential quantification), the variable `l` is only visible inside the constraint and, therefore, it can be bound to different values in different branches. Hence this expression evaluates to the result `5`.

In order to ensure that an encapsulated search will not be suspended due to necessary bindings of free variables, the search goal should be a closed expression when a search operator is applied to it, i.e., the search variable is bound by the lambda abstraction and all other free variables are existentially quantified by local declarations.

8.3 Predefined Search Operators

There are a number of search operators which are predefined in the prelude. All these operators are based on the primitive `try` as described above. It is also possible to define other search strategies in a similar way. Thus, the `try` operator is a a powerful primitive to define appropriate search strategies. In the following, we list the predefined search operators.

`solveAll :: (a->Success) -> [a->Success]`
: Compute all solutions for a search goal via a depth-first search strategy. If there is no solution and the search space is finite, the result is the empty list, otherwise the list contains solved search goals (i.e., without defined operations).

`once :: (a->Success) -> (a->Success)`
: Compute the first solution for a search goal via a depth-first search strategy. Note that `once` is partially defined, i.e., if there is no solution and the search space is finite, the result is undefined.

`findall :: (a->Success) -> [a]`
: Compute all solutions for a search goal via a depth-first search strategy and unpack the solution's values for the search variable into a list.

`best :: (a->Success) -> (a->a->Bool) -> [a->Success]`
: Compute the best solution via a depth-first search strategy, according to a specified relation (the second argument) that can always take a decision which of two solutions is better (the relation should deliver `True` if the first argument is a better solution than the second argument).

  As a trivial example, consider the relation `shorter` defined by

  ```
  shorter l1 l2 = length l1 <= length l2
  ```

Then the expression

```
best (\x -> let y free in append x y =:= [1,2,3]) shorter
```

computes the shortest list which can be obtained by splitting the list `[1,2,3]` into this list and some other list, i.e., it reduces to `[\x->x=:=[]]`. Similarly, the expression

```
best (\x -> let y free in append x y =:= [1,2,3])
     (\l1 l2 -> length l1 > length l2)
```

reduces to `[\x->x=:=[1,2,3]]`.

`one :: (a->Success) -> [a->Success]`
: Compute one solution via a fair strategy. If there is no solution and the search space is finite, the result is the empty list, otherwise the list contains one element (the first solution represented as a search goal).

`condSearch :: (a->Success) -> (a->Bool) -> [a->Success]`
: Compute the first solution (via a fair strategy) that meets a specified condition. The condition (second argument) must be a unary Boolean function.

`browse :: (a->Success) -> IO ()`
: Show the solution of a *solved* constraint on the standard output, i.e., a call `browse g`, where `g` is a solved search goal, is evaluated to an I/O action which prints the solution. If `g` is not an abstraction of a solved constraint, a call `browse g` produces a runtime error.

`browseList :: [a->Success] -> IO ()`
: Similar to `browse` but shows a list of solutions on the standard output. The `browse` operators are useful for testing search strategies. For instance, the evaluation of the expression

```
browseList (solveAll \x -> let y free in append x y =:= [0,1,2])
```

produces the following result on the standard output:

```
[]
[0]
[0,1]
[0,1,2]
```

Due to the laziness of the evaluation strategy, one can also browse the solutions of a goal with infinitely many solutions which are printed on the standard output until the process is stopped.

8.4 Choice

In order to provide a "don't care" selection of an element out of different alternatives, which is necessary in concurrent computation structures to support many-to-one communication, Curry provides the special evaluation annotation `choice`:

f `eval choice`

Intuitively, a function f declared as a `choice` function behaves as follows. A call to f is evaluated as usual (with a fair evaluation of disjunctive alternatives, which is important here!) but with the following differences:

1. No free variable of this function call is bound (i.e., this function call and all its subcalls are considered as rigid).
2. If one rule for f matches and its guard is entailed (i.e., satisfiable without binding goal variables), all other alternatives for evaluating this call are ignored.

This means that the first applicable rule for this function call (after pattern matching and guard checking) is taken and all other alternatives are discarded.

As an example, consider the non-deterministic function `merge` for the fair merge of two lists:

```
merge          :: [a] -> [a] -> [a]
merge          eval choice
merge [] l2    = l2
merge l1 []    = l1
merge (e:r) l2 = e : merge r l2
merge l1 (e:r) = e : merge l1 r
```

Due to the `choice` annotation, a call to `merge` is reducible if one of the input lists is known to be empty or non-empty (but it is not reducible of both inputs are unknown). Thus, `merge` can also merge partially unknown lists which are incrementally instantiated during computation.

Another application of the `choice` annotation is a fair evaluation of disjunctive search goals. For instance, the following function takes a search goal and computes one result (if it exists) for the search variable in a fair manner, i.e., possible infinite computation branches in a disjunction do not inhibit the computation of a solution in another branch:

```
tryone         :: (a->Success) -> a
tryone         eval choice
tryone g | g x = x  where x free
```

Note that functions containing `choice` expressions may be indeterministic in contrast to other user-defined functions, i.e., identical calls may lead to different answers at different times. This is similar to non-deterministic functions which can be given a declarative semantics [18]. The difference to non-deterministic functions is that only one result is computed for functions declared with `choice` instead of all results. As a consequence, the completeness result for Curry's operational semantics [23] does not hold in the presence of the evaluation annotation `choice`. Since indeterministic functions are mainly used in the coordination level of concurrent applications (as in parallel functional computation models [11, 12]), this is a only minor problem. A programming environment for Curry could classify the indeterministic functions in a program.

9 Interface to External Functions and Constraint Solvers

Since Curry has only a small number of builtins, it provides a simple interface to connect external functions (functions programmed in another language) and external constraint solvers. External

functions must be free of side-effects in order to be compatible with Curry's computation model. An external constraint solver consists of a constraint store which can be accessed and manipulated via a few primitive operations.

9.1 External Functions

External functions are considered as an implementation of a potentially infinite set of equations (describing the graph of the functions). In particular, they have no side effects, i.e., identical function calls at different points of time yield identical results. Since the implementation of an external function is unknown, a call to an external function is suspended until all arguments are fully known, i.e., until they are evaluated to ground data terms. This view is compatible with the residuation principle covered by Curry's computation model and very similar to the connection of external functions to logic programs [9, 10].

An external function is declared by a type declaration followed by `external(FName,OF,Lang)` where `FName` is the name of the external function where its code is contained in the object file `OF`. `Lang` is the implementation language of the external function. For instance, if the addition and multiplication on integers are defined as C functions named `add` and `mult` where the compiled code is contained in the file "`arith.o`", we can connect these functions to a Curry program by providing the declarations

```
add  :: Int -> Int -> Int  external("add","arith.o",C)
mult :: Int -> Int -> Int  external("mult","arith.o",C)
```

Implementations of Curry might have restrictions on the interface. A reasonable requirement is that the implementation language is constrained to C and the argument and result types are only simple types like `Bool`, `Int`, `Float`, `Char`, or `String`.

9.2 External Constraint Solvers

A constraint solver can be viewed as an abstract datatype consisting of a constraint store together with a few operations to check the entailment of a constraint or to add a new constraint. In order to connect a constraint solver to Curry, like a solver for arithmetic or Boolean constraints, the external solver must provide the following operations (cs denotes the type of the constraint store and c the type of constraints):

$$
\begin{array}{llll}
new & : & \rightarrow cs & (\textit{create and initialize a new constraint store}) \\
ask & : cs \times c & \rightarrow \{True, False, Unknown\} & (\textit{check entailment of a constraint}) \\
tell & : cs \times c & \rightarrow cs & (\textit{add a new constraint}) \\
clone & : cs & \rightarrow cs \times cs & (\textit{copy the constraint store})
\end{array}
$$

Using these operations, it is relatively easy to extend Curry's computation model to include constraints by adding the constraint store as a new component to the substitution part in answer expressions (cf. Section 3).[12]

[12] This section will be modified or extended in a later version.

10 Literate Programming

To encourage the use of comments for the documentation of programs, Curry supports a literate programming style (inspired by Donald Knuth's "literate programming") where comments are the default case. Non-comment lines containing program code must start with "`>`" followed by a blank. Using this style, we can define the concatenation and reverse function on lists by the following literate program:

```
The following function defines the concatenation of two lists.
Note that this function is predefined as '++' in the standard prelude.

> append :: [t] -> [t] -> [t]
> append []     x  = x
> append (x:xs) ys = x : append xs ys

As a second example for list operations, we define a
function to reverse the order of elements in a list:

> rev :: [t] -> [t]
> rev []     = []
> rev (x:xs) = append (rev xs) [x]
```

To distinguish literature from non-literate programs, literate programs must be stored in files with the extension "`.lcurry`" whereas non-literate programs are stored in files with the extension "`.curry`".

11 Interactive Programming Environment

In order to support different implementations with a comparable user interface, the following commands should be supported by each interactive programming environment for Curry (these commands can also be abbreviated to `:`c where c is the first character of the command):

`:load file` Load the program stored in `file.curry`.

`:reload` Repeat the last load command.

`expr` Evaluate the expression `expr` and show all computed results. Since an expression could be evaluated to a disjunctive expression (cf. Section 3), the expression could be automatically wrapped in some search operator like depth-first search or a fair breadth-first search, depending on the implementation.

`:debug expr` Debug the evaluation of the expression `expr`, i.e., show the single computation steps and ask the user what to do after each single step (like proceed, abort, etc.).

`:type expr` Show the type of the expression `expr`.

`:eval f` Show the definitional tree (see Section D.1) of function `f`.

`:quit` Exit the system.

A Example Programs

This section contains a few example programs together with some initial expressions and their computed results.

A.1 Operations on Lists

Here are simple operations on lists. Note that, due to the logic programming features of Curry, `append` can be used to split lists. We exploit this property to define the last element of a list in a very simple way.

```
-- Concatenation of two lists:
append :: [t] -> [t] -> [t]
append []     ys = ys
append (x:xs) ys = x:append xs ys

-- Naive reverse of all list elements:
rev :: [t] -> [t]
rev []     = []
rev (x:xs) = append (rev xs) [x]

-- Last element of a list:
last :: [t] -> t
last xs | append _ [x] =:= xs  = x  where x free
```

Expressions and their evaluated results:

```
append [0,1] [2,3]   ⇝   [0,1,2,3]

append L M =:= [0,1]
  ⇝   {L=[],M=[0,1]} | {L=[0],M=[1]} | {L=[0,1],M=[]}

rev [0,1,2,3]   ⇝   [3,2,1,0]

last (append [1,2] [3,4])   ⇝   4
```

A.2 Higher-Order Functions

Here are some "traditional" higher-order functions to show that the familiar functional programming techniques can be applied in Curry. Note that the functions `map`, `foldr`, and `filter` are predefined in Curry (see Appendix B).

```
-- Map a function on a list (predefined):
map :: (t1->t2) -> [t1] -> [t2]
map f []     = []
map f (x:xs) = f x:map f xs

-- Fold a list (predefined):
foldr :: (t1->t2->t2) -> t2 -> [t1] -> t2
foldr f a []     = a
foldr f a (x:xs) = f x (foldr f a xs)

-- Filter elements in a list (predefined):
filter :: (t -> Bool) -> [t] -> [t]
filter p []     = []
filter p (x:xs) = if p x then x:filter p xs
                         else filter p xs

-- Quicksort function:
quicksort :: [Int] -> [Int]
quicksort []      = []
quicksort (x:xs)  = quicksort (filter (<= x) xs)
                    ++ [x]
                    ++ quicksort (filter (> x) xs)
```

A.3 Relational Programming

Here is a traditional example from logic programming: a simple deductive database with family relationships. We use a relational programming style, i.e., all relationships are represented as constraints (i.e., functions with result type `Success`).

```
-- Declaration of an enumeration type for persons:
-- (as an alternative, one could consider persons as strings)

data Person = Christine | Maria | Monica | Alice | Susan |
              Antony | Bill | John | Frank | Peter | Andrew

-- Two basic relationships:

married :: Person -> Person -> Success
married Christine Antony  = success
married Maria Bill        = success
married Monica John       = success
married Alice Frank       = success

mother :: Person -> Person -> Success
mother Christine John   = success
mother Christine Alice  = success
mother Maria Frank      = success
mother Monica Susan     = success
mother Monica Peter     = success
mother Alice Andrew     = success

-- and here are the deduced relationships:
father :: Person -> Person -> Success
father f c = let m free in married m f & mother m c

grandfather :: Person -> Person -> Success
grandfather g c = let f free in father g f & father f c
grandfather g c = let m free in father g m & mother m c
```

Expressions and their evaluated results:

```
father John child   ⇝   {child=Susan} | {child=Peter}

grandfather g c   ⇝
{g=Antony,c=Susan} | {g=Antony,c=Peter} |
{g=Bill,c=Andrew}  | {g=Antony,c=Andrew}
```

A.4 Functional Logic Programming

This is the same example as in the previous section. However, we use here a functional logic programming style which is more readable but provides the same goal solving capabilities. The basic functions are `husband` and `mother` which express the functional dependencies between the different persons. Note that the derived function `grandfather` is a non-deterministic function which yields all grandfathers for a given person.

```
data Person = Christine | Maria | Monica | Alice | Susan |
              Antony | Bill | John | Frank | Peter | Andrew

-- Two basic functional dependencies:

husband :: Person -> Person
husband Christine  = Antony
husband Maria      = Bill
husband Monica     = John
husband Alice      = Frank

mother :: Person -> Person
mother John    = Christine
mother Alice   = Christine
mother Frank   = Maria
mother Susan   = Monica
mother Peter   = Monica
mother Andrew  = Alice

-- and here are the deduced functions and relationships:
father :: Person -> Person
father c  = husband (mother c)

grandfather :: Person -> Person
grandfather g = father (father g)
grandfather g = father (mother g)
```

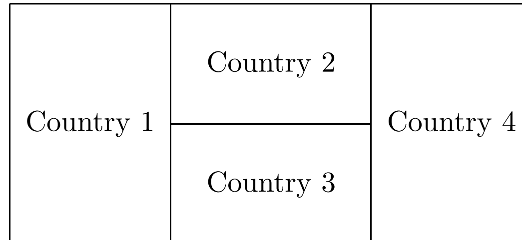
Expressions and their evaluated results:

```
father Child =:= john   ⇝   {Child=susan} | {Child=peter}

grandfather c   ⇝
{c=Susan} Antony | {c=Peter} Antony | {c=Andrew} Bill | {c=Andrew} Antony
```

A.5 Constraint Solving and Concurrent Programming

In this example we demonstrate how Curry's concurrent features can be used to solve constraint problems with finite domains more efficiently than a simple generate-and-test solver. We want to solve the classical map coloring problem. Consider the following simple map:



The problem is to assign to each of the four countries a color (red, green, yellow, or blue) such that countries with a common border have different colors.

To solve this problem, we have to specify that some countries have different colors. For this purpose, we define the following constraint which checks to elements for incompatibility:

```
diff :: a -> a -> Success
diff x y = (x==y)=:=False
```

For instance, `diff` t_1 t_2 is satisfied if t_1 and t_2 are different constants. If one of the arguments is a variable, the evaluation of the function is delayed by definition of `==`.

Now there is a straightforward solution of the map coloring problem. We define a constraint `coloring` specifying the valid colors for each country and a constraint `correct` specifying which countries must have different colors:

```
data Color = Red | Green | Yellow | Blue

isColor :: Color -> Success
isColor Red    = success
isColor Yellow = success
isColor Green  = success
isColor Blue   = success

coloring :: Color -> Color -> Color -> Color -> Success
coloring l1 l2 l3 l4 = isColor l1 & isColor l2 & isColor l3 & isColor l4

correct :: Color -> Color -> Color -> Color -> Success
correct l1 l2 l3 l4 =
         diff l1 l2 & diff l1 l3 & diff l2 l3 & diff l2 l4 & diff l3 l4
```

In classical logic programming, we can compute the solutions to the map coloring problem by enumerating all potential solutions followed by a check whether a potential solution is a correct

one ("generate and test"). This can be expressed by solving the following goal:[13]

```
coloring l1 l2 l3 l4 & correct l1 l2 l3 l4
```

However, a much faster solution can be obtained by reversing the order of the tester and the generator:

```
correct l1 l2 l3 l4 & coloring l1 l2 l3 l4
```

The latter constraint is evaluated in a concurrent way. In the first steps, the subexpression `correct l1 l2 l3 l4` is reduced to the constraint

```
diff l1 l2 & diff l1 l3 & diff l2 l3 & diff l2 l4 & diff l3 l4
```

which is then reduced to

```
(l1==l2)=:=False & (l1==l3)=:=False & (l2==l3)=:=False & (l2==l4)=:=False
 & (l3==l4)=:=False
```

This constraint cannot be further evaluated since the arguments to `==` are free variables. Therefore, it is suspended and the final equational constraint `coloring l1 l2 l3 l4` is evaluated which binds the variables to the potential colors. Since the variable binding is performed by consecutive computation steps, the equalities are evaluated as soon as their arguments are bound. For instance, if the variables `l1` and `l2` are bound to the color `Red`, the first constraint `(l1==l2)=:=False` cannot be solved (due to the unsolvability of the equational constraint `True=:=False`) which causes the failure of the entire goal. As a consequence, not all potential colorings are computed (as in the generate-and-test approach) but only those colorings for which the constraints `correct l1 l2 l3 l4` is satisfiable. Therefore, the (dis)equality goals act as "passive constraints" aiming to reduce the search space.

[13]We assume an implementation that processes the concurrent conjunction c_1 & c_2 in a "sequential-first" manner: first c_1 is solved and, if this is not possible due to a suspended evaluation, c_2 is solved.

A.6 Concurrent Object-Oriented Programming

The following example shows a simple method to program in a concurrent object-oriented style in Curry. For this purpose, an object is a process waiting for incoming messages. The local state is a parameter of this process. Thus, a process is a function of type

```
State -> [Message] -> Success
```

In the subsequent example, we implement a bank account as an object waiting for messages of the form `Deposit i`, `Withdraw i`, or `Balance i`. Thus, the bank account can be defined as follows:

```
data Message = Deposit Int | Withdraw Int | Balance Int

account :: Int -> [Message] -> Success
account eval rigid  -- account is a consumer
account _ []                  =  success
account n (Deposit  a : ms)  =  account (n+a) ms
account n (Withdraw a : ms)  =  account (n-a) ms
account n (Balance  b : ms)  =  b=:=n & account n ms

-- Install an initial account with message stream s:
make_account s = account 0 s
```

A new account object is created by the constraint `make_account s` where `s` is a free variable. When `s` is instantiated with messages, the account process starts processing these messages. The following concurrent conjunction of constraints creates an account and sends messages:

```
make_account s & s=:=[Deposit 200, Deposit 50, Balance b]
```

After this goal is solved, the free variable `b` has been bound to `250` representing the balance after the two deposits.

To show a client-server interaction, we define a client of a bank account who is finished if his account is `50`, buys things for `30` if his account is greater than `50`, and works for an income of `70` if his account is less than `50`. In order to get the client program independent of the account processing, the client sends messages to his account. Therefore, the client is implemented as follows:

```
-- Send a message to an object identified by its message stream obj:
sendMsg :: [msg] -> msg -> [msg]
sendMsg msg obj | obj =:= msg:obj1  = obj1  where obj1 free

-- Client process of an bank account
client s | s1 =:= sendMsg (Balance b) s  =
  if b==50 then s1=:=[]                                   -- stop
           else if b>50 then client (sendMsg (Withdraw 30) s1)  -- buy
                        else client (sendMsg (Deposit  70)  s1) -- work
  where s1,b free
```

We start the account and client process with an initial amount of `100`:

```
make_account s & client (sendMsg (Deposit 100) s)
```

A Curry system evaluates this goal by a concurrent evaluation of both processes and computes the final answer

```
{s=[Deposit 100, Balance 100, Withdraw 30, Balance 70, Withdraw 30,
    Balance 40, Deposit 70, Balance 110, Withdraw 30, Balance 80,
    Withdraw 30, Balance 50]}
```

which shows the different messages sent to the account process.

B Standard Prelude

This section contains the standard prelude for Curry programs.[14] This prelude will be imported to every Curry program, i.e., the names declared in the prelude cannot be redefined.

```
module prelude where

-- Infix operator declarations

infixl 9 !!
infixr 9 .
infixl 7 *, /, `div`, `mod`
infixl 6 +, -
infixr 5 ++, :
infix  4 =:=, ==, /=, <, >, <=, >=
infix  4  `elem`, `notElem`
infixr 3 &&
infixr 2 ||
infixl 1 >>, >>=
infixr 0 $, $!, `seq`, &, &>


-- Some standard combinators:

-- Function composition
(.)             :: (b -> c) -> (a -> b) -> (a -> c)
f . g           = \x -> f (g x)

-- Identity
id              :: a -> a
id x            = x

-- Constant function
const           :: a -> b -> a
const x _       = x

-- Convert an uncurried function to a curried function
curry           :: ((a,b) -> c) -> a -> b -> c
curry f a b     =  f (a,b)

-- Convert an curried function to a function on pairs
uncurry         :: (a -> b -> c) -> (a,b) -> c
uncurry f (a,b) = f a b
```

[14]The definition will be extended in a later version.

```
-- (flip f) is identical to f but with the order of arguments reversed
flip            :: (a -> b -> c) -> b -> a -> c
flip  f x y     = f y x

-- Repeat application of a function until a predicate holds
until          :: (a -> Bool) -> (a -> a) -> a -> a
until p f x     = if p x then x else until p f (f x)

-- Right-associative application
($)             :: (a -> b) -> a -> b
f $ x           = f x

-- Evaluate the first argument to head normal form and return the
-- second argument. Suspend if the first argument evaluates to a
-- free variable.
seq     :: a -> b -> b

-- Right-associative application with strict evaluation of its argument.
($!)    :: (a -> b) -> a -> b
f $! x  = x `seq` f x

-- Abort the execution with an error message
error :: String -> _

-- failed is a non-reducible polymorphic function.
-- It is useful to express a failure in a search branch of the execution.
failed :: _
failed | 1=:=2 = x where x free


-- Boolean values

data Bool = True | False

-- Sequential conjunction
(&&)                   :: Bool -> Bool -> Bool
True  && x             = x
False && _             = False

-- Sequential disjunction
(||)                   :: Bool -> Bool -> Bool
True  || _             = True
False || x             = x

-- Negation
```

```
not                     :: Bool -> Bool
not True                = False
not False               = True

-- Conditional
if_then_else            :: Bool -> a -> a -> a
if_then_else True  t _ = t
if_then_else False _ f = f

otherwise               :: Bool
otherwise               = True

-- Disequality
(/=)                    :: a -> a -> Bool
x /= y                  = not (x==y)


-- Pairs

data (a,b) = (a,b)

fst                :: (a,b) -> a
fst (x,_)          = x

snd                :: (a,b) -> b
snd (_,y)          = y


-- Unit type

data () = ()


-- Lists

data [a] = [] | a : [a]

-- First element of a list
head               :: [a] -> a
head (x:_)         = x

-- Remaining elements of a list
tail               :: [a] -> [a]
tail (_:xs)        = xs
```

```
-- Is a list empty?
null            :: [_] -> Bool
null []         = True
null (_:_)      = False

-- Concatenation
(++)              :: [a] -> [a] -> [a]
[]     ++ ys      = ys
(x:xs) ++ ys      = x : xs++ys

-- List length
length            :: [a] -> Int
length []         = 0
length (_:xs)     = 1 + length xs

-- List index (subscript) operator, head has index 0
(!!)              :: [a] -> Int -> a
(x:xs) !! n | n==0      = x
            | n>0       = xs !! (n-1)

-- Map a function on a list
map               :: (a->b) -> [a] -> [b]
map _ []          = []
map f (x:xs)      = f x : map f xs

-- Accumulate all list elements by applying a binary operator from
-- left to right, i.e.,
-- foldl f z [x1,x2,...,xn] = (...((z `f` x1) `f` x2) ...) `f` xn :
foldl            :: (a -> b -> a) -> a -> [b] -> a
foldl _ z []     = z
foldl f z (x:xs) = foldl f (f z x) xs

-- Accumulate a non-empty list from left to right:
foldl1           :: (a -> a -> a) -> [a] -> a
foldl1 f (x:xs)  = foldl f x xs

-- Accumulate all list elements by applying a binary operator from
-- right to left, i.e.,
-- foldr f z [x1,x2,...,xn] = (x1 `f` (x2 `f` ... (xn `f` z)...)) :
foldr             :: (a->b->b) -> b -> [a] -> b
foldr _ z []      = z
foldr f z (x:xs)  = f x (foldr f z xs)

-- Accumulate a non-empty list from right to left:
foldr1              :: (a -> a -> a) -> [a] -> a
```

```
foldr1 _ [x]        = x
foldr1 f (x1:x2:xs) = f x1 (foldr1 f (x2:xs))

-- Filter elements in a list
filter            :: (a -> Bool) -> [a] -> [a]
filter _ []       = []
filter p (x:xs)   = if p x then x : filter p xs
                             else filter p xs

-- Join two lists into one list of pairs. If one input list is shorter than
-- the other, the additional elements of the longer list are discarded.
zip               :: [a] -> [b] -> [(a,b)]
zip []     _      = []
zip (_:_)  []     = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys

-- Join three lists into one list of triples. If one input list is shorter than
-- the other, the additional elements of the longer lists are discarded.
zip3                      :: [a] -> [b] -> [c] -> [(a,b,c)]
zip3 []     _      _      = []
zip3 (_:_)  []     _      = []
zip3 (_:_)  (_:_)  []     = []
zip3 (x:xs) (y:ys) (z:zs) = (x,y,z) : zip3 xs ys zs

-- Join two lists into one list by applying a combination function to
-- corresponding pairs of elements (i.e., zip = zipWith (,)):
zipWith                 :: (a->b->c) -> [a] -> [b] -> [c]
zipWith _ []     _      = []
zipWith _ (_:_)  []     = []
zipWith f (x:xs) (y:ys) = f x y : zipWith f xs ys

-- Join three lists into one list by applying a combination function to
-- corresponding triples of elements (i.e., zip3 = zipWith3 (,,)):
zipWith3                        :: (a->b->c->d) -> [a] -> [b] -> [c] -> [d]
zipWith3 _ []     _      _      = []
zipWith3 _ (_:_)  []     _      = []
zipWith3 _ (_:_)  (_:_)  []     = []
zipWith3 f (x:xs) (y:ys) (z:zs) = f x y z : zipWith3 f xs ys zs

-- Transform a list of pairs into a pair of lists
unzip               :: [(a,b)] -> ([a],[b])
unzip []            = ([],[])
unzip ((x,y):ps)    = (x:xs,y:ys) where (xs,ys) = unzip ps

-- Transform a list of triples into a triple of lists
```

```
unzip3              :: [(a,b,c)] -> ([a],[b],[c])
unzip3 []           = ([],[],[])
unzip3 ((x,y,z):ts) = (x:xs,y:ys,z:zs) where (xs,ys,zs) = unzip3 ts

-- Concatenate a list of lists into one list
concat             :: [[a]] -> [a]
concat l           = foldr (++) [] l

-- Map a function from elements to lists and merge the result into one list
concatMap          :: (a -> [b]) -> [a] -> [b]
concatMap f        = concat . map f

-- Infinite list of repeated applications of a function f to an element x:
-- iterate f x = [x, f x, f (f x),...]
iterate            :: (a -> a) -> a -> [a]
iterate f x        = x : iterate f (f x)

-- Infinite list where all elements have the same value x:
repeat             :: a -> [a]
repeat x           = x : repeat x

-- List of length n where all elements have the same value x:
replicate          :: Int -> a -> [a]
replicate n x      = take n (repeat x)

-- Return prefix of length n
take               :: Int -> [a] -> [a]
take n l           = if n==0 then [] else takep n l
   where takep _ []     = []
         takep n (x:xs) = x : take (n-1) xs

-- Return suffix without first n elements
drop               :: Int -> [a] -> [a]
drop n l           = if n==0 then l else dropp n l
   where dropp _ []     = []
         dropp n (_:xs) = drop (n-1) xs

-- (splitAt n xs) is equivalent to (take n xs, drop n xs)
splitAt            :: Int -> [a] -> ([a],[a])
splitAt n l        = if n==0 then ([],l) else splitAtp n l
   where splitAtp _ []     = ([],[])
         splitAtp n (x:xs) = let (ys,zs) = splitAt (n-1) xs in (x:ys,zs)

-- Return longest prefix with elements satisfying a predicate
takeWhile           :: (a -> Bool) -> [a] -> [a]
```

```
takeWhile _ []     = []
takeWhile p (x:xs) = if p x then x : takeWhile p xs else []

-- Return suffix without takeWhile prefix
dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile _ []     = []
dropWhile p (x:xs) = if p x then dropWhile p xs else x:xs

-- (span p xs) is equivalent to (takeWhile p xs, dropWhile p xs)
span               :: (a -> Bool) -> [a] -> ([a],[a])
span _ []          = ([],[])
span p (x:xs)
       | p x       = let (ys,zs) = span p xs in (x:ys, zs)
       | otherwise = ([],x:xs)

-- (break p xs) is equivalent to (takeWhile (not.p) xs, dropWhile (not.p) xs)
-- i.e., it breaks a list at the first occurrence of an element satisfying p
break              :: (a -> Bool) -> [a] -> ([a],[a])
break p            = span (not . p)

-- Break a string into list of lines where a line is terminated at a
-- newline character. The resulting lines do not contain newline characters.
lines :: String -> [String]
lines [] = []
lines (c:cs) = let (l,restcs) = splitline (c:cs) in l : lines restcs
 where splitline []     = ([],[])
       splitline (c:cs) = if c=='\n'
                          then ([],cs)
                          else let (ds,es) = splitline cs in (c:ds,es)

-- Concatenate a list of strings with terminating newlines
unlines    :: [String] -> String
unlines ls = concatMap (++"\n") ls

-- Break a string into a list of words where the words are delimited by
-- white spaces.
words      :: String -> [String]
words s    = let s1 = dropWhile isSpace s
              in if s1=="" then []
                           else let (w,s2) = break isSpace s1
                                 in w : words s2
 where
    isSpace c = c == ' '  || c == '\t' || c == '\n' || c == '\r'

-- Concatenate a list of strings with a blank between two strings.
```

```
unwords    :: [String] -> String
unwords ws = if ws==[] then []
                       else foldr1 (\w s -> w ++ ' ':s) ws

-- Reverse the order of all elements in a list
reverse    :: [a] -> [a]
reverse    = foldl (flip (:)) []

-- Compute the conjunction of a Boolean list
and        :: [Bool] -> Bool
and        = foldr (&&) True

-- Compute the disjunction of a Boolean list
or         :: [Bool] -> Bool
or         = foldr (||) False

-- Is there an element in a list satisfying a given predicate?
any        :: (a -> Bool) -> [a] -> Bool
any p      = or . map p

-- Is a given predicate satisfied by all elements in a list?
all        :: (a -> Bool) -> [a] -> Bool
all p      = and . map p

-- Element of a list?
elem       :: a -> [a] -> Bool
elem x     = any (x==)

-- Not element of a list?
notElem    :: a -> [a] -> Bool
notElem x  = all (x/=)

--- Looks up a key in an association list.
lookup            :: a -> [(a,b)] -> Maybe b
lookup _ []       = Nothing
lookup k ((x,y):xys)
      | k==x      = Just y
      | otherwise = lookup k xys


-- Generating arithmetic sequences:
enumFrom               :: Int -> [Int]                -- [n..]
enumFrom n             = n : enumFrom (n+1)

enumFromThen           :: Int -> Int -> [Int]         -- [n1,n2..]
```

```
enumFromThen n1 n2     = iterate ((n2-n1)+) n1

enumFromTo             :: Int -> Int -> [Int]            -- [n..m]
enumFromTo n m         = if n>m then [] else n : enumFromTo (n+1) m

enumFromThenTo         :: Int -> Int -> Int -> [Int]     -- [n1,n2..m]
enumFromThenTo n1 n2 m = takeWhile p (enumFromThen n1 n2)
                           where p x | n2 >= n1  = (x <= m)
                                     | otherwise = (x >= m)


-- Conversion functions between characters and their ASCII values

ord :: Char -> Int
chr :: Int -> Char


-- Convert a term into a printable representation

show :: a -> String


-- Types of primitive arithmetic functions and predicates

(+)   :: Int -> Int -> Int
(-)   :: Int -> Int -> Int
(*)   :: Int -> Int -> Int
div   :: Int -> Int -> Int
mod   :: Int -> Int -> Int
(<)   :: Int -> Int -> Bool
(>)   :: Int -> Int -> Bool
(<=)  :: Int -> Int -> Bool
(>=)  :: Int -> Int -> Bool


-- Constraints

-- Equational constraint
(=:=) :: a -> a -> Success

-- Always solvable constraint
success :: Success

-- Concurrent conjunction of constraints
(&)  :: Success -> Success -> Success
```

```
-- Sequential conjunction of constraints
(&>) :: Success -> Success -> Success
c1 &> c2 | c1 = c2


-- Maybe type

data Maybe a = Nothing | Just a

maybe              :: b -> (a -> b) -> Maybe a -> b
maybe n _ Nothing  = n
maybe _ f (Just x) = f x


-- Either type

data Either a b = Left a | Right b

either               :: (a -> c) -> (b -> c) -> Either a b -> c
either f _ (Left x)  = f x
either _ g (Right x) = g x


-- Monadic IO

data IO a  -- conceptually: World -> (a,World)

(>>=)             :: IO a -> (a -> IO b) -> IO b
return            :: a -> IO a

(>>)              :: IO a -> IO b -> IO b
a >> b            = a >>= (\_ -> b)

done              :: IO ()
done              = return ()

putChar           :: Char -> IO ()
getChar           :: IO Char

readFile          :: String -> IO String
writeFile         :: String -> String -> IO ()
appendFile        :: String -> String -> IO ()

putStr            :: String -> IO ()
```

```
putStr []          = done
putStr (c:cs)      = putChar c >> putStr cs

putStrLn           :: String -> IO ()
putStrLn cs        = putStr cs >> putChar '\n'

getLine            :: IO String
getLine            = do c <- getChar
                        if c=='\n' then return []
                                   else do cs <- getLine
                                           return (c:cs)

-- Convert a term into a string and print it
print              :: _ -> IO ()
print t            = putStrLn (show t)

-- Solve a constraint as an I/O action:
-- Note: the constraint should be always solvable in a deterministic way
doSolve :: Success -> IO ()
doSolve constraint | constraint = done


-- IO monad auxiliary functions:

-- Execute a sequence of I/O actions and collect all results in a list:
sequenceIO        :: [IO a] -> IO [a]
sequenceIO []      = return []
sequenceIO (c:cs) = do x  <- c
                        xs <- sequenceIO cs
                        return (x:xs)

-- Execute a sequence of I/O actions and ignore the results:
sequenceIO_         :: [IO _] -> IO ()
sequenceIO_          = foldr (>>) done

-- Map an I/O action function on a list of elements.
-- The results of all I/O actions are collected in a list.
mapIO              :: (a -> IO b) -> [a] -> IO [b]
mapIO f             = sequenceIO . map f

-- Map an I/O action function on a list of elements.
-- The results of all I/O actions are ignored.
mapIO_             :: (a -> IO _) -> [a] -> IO ()
mapIO_ f            = sequenceIO_ . map f
```

```
-- Encapsulated search

-- primitive operator to control non-determinism
try :: (a->Success) -> [a->Success]

-- compute all solutions via depth-first search
solveAll      :: (a->Success) -> [a->Success]
solveAll g    = collect (try g)
              where collect []         = []
                    collect [g]        = [g]
                    collect (g1:g2:gs) = concat (map solveAll (g1:g2:gs))

-- compute first solution via depth-first search
once          :: (a->Success) -> (a->Success)
once g        = head (solveAll g)

-- compute all values of solutions via depth-first search
findall       :: (a->Success) -> [a]
findall g     = map unpack (solveAll g)

-- compute best solution via branch-and-bound with depth-first search
best          :: (a->Success) -> (a->a->Bool) -> [a->Success]

-- try to find a solution to a search goal via a fair search
-- (and fail if there is no solution)
tryone         :: (a->Success) -> a
tryone         eval choice
tryone g | g x = x  where x free

-- compute one solution with a fair search strategy
one            :: (a->Success) -> [a->Success]
one g          = solveAll \x -> x =:= tryone g

-- compute one solution with a fair search satisfying a condition
condSearch        :: (a->Success) -> (a->Bool) -> [a->Success]
condSearch g cond = one \x -> g x & cond x =:= True

-- show the solution of a solved constraint
browse        :: (a->Success) -> IO ()
browse g      = putStr (show (unpack g))

-- show the solutions of a list of solved constraints
browseList        :: [a->Success] -> IO ()
browseList []     = done
```

```
browseList (g:gs) = browse g >> putChar '\n' >> browseList gs


-- unpack a solution's value from a (solved) search goal
unpack          :: (a -> Success) -> a
unpack g | g x  = x  where x free
```

C Syntax of Curry

The syntax is still preliminary and not intended to be used by automatic tools. Nevertheless, it provides a precise definition of the concrete syntax which can be further discussed.

The syntax is close to Haskell but the following differences should be noted.

- As-patterns are missing
- Case expressions are missing
- Currently, there are no infix constructors except for "`:`", the predefined list *cons*tructor. They will be added later, although they are already used in concrete example programs.

C.1 Lexicon

The case of identifiers matters, i.e., "abc" differs from "Abc". There are four *case modes* selectable at compilation time:

- Prolog mode: variables start with an upper case latter, all other identifier symbols start with a lower case letter.
- Gödel mode: like Prolog mode with the cases swapped.
- Haskell mode: see section 1.3 of the Haskell report.
- free: no constraints on the case of identifiers.

The default case mode is *free*. If a case mode is selected but not obeyed, the compiler issues a warning.

The syntax does not define the following non-terminal symbols defining classes of identifiers: *ModuleID*, *TypeConstrID*, *DataConstrID*, *TypeVarID*, *InfixOpID*, *FunctionID*, *VariableID*. All, except *InfixOpID*, consist of an initial letter, whose upper or lower case depend on the case mode, followed by any number of letters, digits, underscores, and single quotes. Additionally, *ModuleID* can contain dots at inner positions. *InfixOpID* is any string of characters from the string "`~!@#$%^&*+-=<>?./|\:`" or another identifier (like *VariableID*) enclosed in `‘...‘` like `‘mod‘`.

The following symbols are *keywords* and cannot be used as an identifier:

```
as      case    choice  data    do       else    external
free    hiding  if      import  in       infix   infixl
infixr  let     module  of      qualified rigid  then
type    where
```

The syntax leaves undefined *Literal* of primitive types. These are literal constants, such as "`1`", "`3.14`", or the character "`'a'`". They are as in Java. Java, in constrast to Haskell, adopts the Unicode standard to represent characters and character strings.

Comments begins either with "`--`" and terminate at the end of the line or with "`{-`" and terminate with a matching "`-}`", i.e., the delimiters "`{-`" and "`-}`" act as parentheses and can be nested.

C.2 Layout

Similarly to Haskell, a Curry programmer can use layout information to define the structure of blocks. For this purpose, we define the indentation of a symbol as the column number indicating the start of this symbol. The indentation of a line is the indentation of its first symbol.[15]

The layout (or "off-side") rule applies to lists of syntactic entities after the keywords `let`, `where`, `do`, or `of`. In the subsequent context-free syntax (Section C.3), these lists are enclosed with curly brackets ({ }) and the single entities are separated by semicolons (`;`). Instead of using the curly brackets and semicolons of the context-free syntax, a Curry programmer must specify these lists by indentation: the indentation of a list of syntactic entities after `let`, `where`, `do`, or `of` is the indentation of the next symbol following the `let`, `where`, `do`, `of`. Any item of this list start with the same indentation as the list. Lines with only whitespaces or an indentation greater than the indentation of the list continue the item in its previous line. Lines with an indentation less than the indentation of the list terminates the entire list. Moreover, a list started by `let` is terminated by the keyword `in`. Thus, the sentence

```
f x = h x where {g y = y+1 ; h z = (g z) * 2 }
```

which is valid w.r.t. the context-free syntax, is written with the layout rules as

```
f x = h x
 where g y = y+1
       h z = (g z) * 2
```

or also as

```
f x = h x  where
  g y = y+1
  h z = (g z)
          * 2
```

To avoid an indentation of top-level declarations, the keyword `module` and the end-of-file token are assumed to start in column 0.

C.3 Context Free Syntax

Module ::= `module` *ModuleID* [*Exports*] `where` *Block*
ModuleID ::= see lexicon
Exports ::= `(` $Export_1$ `,` ... `,` $Export_n$ `)`
Export ::= *QFunctionName*
| *QTypeConstrID*
| *QTypeConstrID* `(..)`
| `module` *ModuleID*
Block ::= { [$ImportDecl_1$; ... ; $ImportDecl_n$;] $(n \geqslant 0)$ [$FixityDeclaration_1$; ... ; $FixityDeclaration_n$;] $(n \geqslant 0)$ $BlockDeclaration_1$; ... ; $BlockDeclaration_m$ } $(m \geqslant 0)$
ImportDecl ::= `import` [`qualified`] *ModuleID* [`as` *ModuleID*] [*ImportRestr*]

[15] In order to determine the exact column number, we assume a fixed-width font with tab stops at each 8th column.

ImportRestr ::= `(` $Import_1$ `,` … `,` $Import_n$ `)`
| `hiding` `(` $Import_1$ `,` … `,` $Import_n$ `)`
Import ::= *FunctionName*
| *TypeConstrID*
| *TypeConstrID* `(..)`
BlockDeclaration ::= *TypeSynonymDecl*
| *DataDeclaration*
| *FunctionDeclaration*
TypeSynonymDecl ::= `type` *TypeConstrID* $TypeVarID_1$ … $TypeVarID_n$ $(n \geqslant 0)$ `=` *TypeExpr*
DataDeclaration ::= `data` *TypeDeclaration*
TypeDeclaration ::= *TypeConstrID* $TypeVarID_1$ … $TypeVarID_n$ $(n \geqslant 0)$ `=`
$ConstrDeclaration_1$ `|` … `|` $ConstrDeclaration_m$ $(m > 0)$
TypeConstrID ::= see lexicon
ConstrDeclaration ::= *DataConstrID* $SimpleTypeExpr_1$ … $SimpleTypeExpr_n$ $(n \geqslant 0)$
DataConstrID ::= see lexicon
TypeExpr ::= *TypeAppl* [`->` *TypeExpr*]
TypeAppl ::= *QTypeConstrID* $SimpleTypeExpr_1$ … $SimpleTypeExpr_n$ $(n \geqslant 0)$
SimpleTypeExpr ::= *QTypeConstrID*
| *TypeVarID* | `_`
| `()`
| `(` *TypeExpr* `)`
| `(` $TypeExpr_1$ `,` … `,` $TypeExpr_n$ `)` $(n > 1)$
| `[` *TypeExpr* `]`
TypeVarID ::= see lexicon
FixityDeclaration ::= *FixityKeyword* *Natural* $InfixOpID_1$ `,` … `,` $InfixOpID_n$ $(n > 0)$
FixityKeyword ::= `infixl` | `infixr` | `infix`
Natural ::= *Digit* | *Digit Natural*
Digit ::= `0` | `1` | `2` | `3` | `4` | `5` | `6` | `7` | `8` | `9`
InfixOpID ::= see lexicon
FunctionDeclaration ::= *Signature* | *EvalAnnot* | *Equat*
Signature ::= *FunctionNames* `::` *TypeExpr*
FunctionNames ::= $FunctionName_1$ `,` … `,` $FunctionName_n$ $(n > 0)$
FunctionName ::= `(` *InfixOpID* `)`
| *FunctionID*
FunctionID ::= see lexicon
EvalAnnot ::= *FunctionNames* `eval` *Annotation*
Annotation ::= `rigid` | `choice`
Equat ::= *FunLHS* `=` *Expr* [`where` *LocalDefs*]
| *FunLHS CondExprs* [`where` *LocalDefs*]
FunLHS ::= *FunctionName* $SimplePattern_1$ … $SimplePattern_n$ $(n \geqslant 0)$
| *SimplePattern InfixOpID SimplePattern*
Pattern ::= *QDataConstrID* $Pattern_1$ … $Pattern_n$ [`:` *Pattern*] $(n > 0)$
| *SimplePattern* [`:` *Pattern*]

SimplePattern ::= *VariableID* | `_`
| *QDataConstrID*
| *Literal*
| `()`
| `(` $Pattern_1$ `,` ... `,` $Pattern_n$ `)` $(n > 1)$
| `(` *Pattern* `)`
| `[` $Pattern_1$ `,` ... `,` $Pattern_n$ `]` $(n \geqslant 0)$
VariableID ::= see lexicon
LocalDefs ::= `{` $ValueDeclaration_1$ `;` ... `;` $ValueDeclaration_n$ `}` $(n > 0)$
ValueDeclaration ::= *FunctionDeclaration*
| *PatternDeclaration*
| $VariableID_1$ `,` ... `,` $VariableID_n$ `free` $(n \geqslant 0)$
PatternDeclaration ::= *Pattern* `=` *Expr* [`where` *LocalDefs*]
CondExprs ::= `|` *Expr* `=` *Expr* [*CondExprs*]
Expr ::= `\` $SimplePattern_1$ ... $SimplePattern_n$ `->` *Expr* $(n > 0)$
| `let` *LocalDefs* `in` *Expr* (*let expression*)
| `if` *Expr* `then` *Expr* `else` *Expr* (*conditional*)
| `case` *Expr* `of` `{` Alt_1 `;` ... `;` Alt_n `}` (*case expression*, $n \geqslant 0$)
| `do` `{` $Stmt_1$ `;` ... `;` $Stmt_n$ `;` *Expr* `}` (*do expression*, $n \geqslant 0$)
| *Expr* *QInfixOpID* *Expr* (*infix operator*)
| `-` *Expr* (*unary minus*)
| *FunctExpr*
FunctExpr ::= [*FunctExpr*] *BasicExpr* (*function application*)
BasicExpr ::= *QVariableID* (*variable*)
| *QDataConstrID* (*data constructor*)
| *QFunctionID* (*defined function*)
| `(` *QInfixOpID* `)` (*operator function*)
| *Literal*
| `()` (*empty tuple*)
| `(` *Expr* `)` (*parenthesized expression*)
| `(` $Expr_1$ `,` ... `,` $Expr_n$ `)` (*tuple*, $n > 1$)
| `[` $Expr_1$ `,` ... `,` $Expr_n$ `]` (*finite list*, $n \geqslant 0$)
| `[` *Expr* [`,` *Expr*] `..` [*Expr*] `]` (*arithmetic sequence*)
| `[` *Expr* `|` $Qual_1$ `,` ... `,` $Qual_n$ `]` (*list comprehension*, $n \geqslant 1$)
| `(` *Expr* *QInfixOpID* `)` (*left section*)
| `(` *QInfixOpID* *Expr* `)` (*right section*)
Alt ::= *Pattern* `->` *Expr*
Qual ::= *Expr*
| *Pattern* `<-` *Expr*
Stmt ::= *Expr*
| `let` *LocalDefs*
| *Pattern* `<-` *Expr*
QTypeConstrID ::= [*ModuleID* `.`] *TypeConstrID*
QDataConstrID ::= [*ModuleID* `.`] *DataConstrID*

QFunctionName	::=	(*QInfixOpID*) | *QFunctionID*
QInfixOpID	::=	[*ModuleID* .] *InfixOpID*
QFunctionID	::=	[*ModuleID* .] *FunctionID*
QVariableID	::=	[*ModuleID* .] *VariableID*
Literal	::=	*Int* | *Char* | *String* | *Float*
Int	::=	see lexicon
Char	::=	see lexicon
String	::=	see lexicon
Float	::=	see lexicon

If the alternative *FunctionDeclaration* is used in a *ValueDeclaration*, then the left-hand side (*FunLHS*) must have at least one pattern after the *FunctionName* (instead of zero patterns which is possible in top-level function declarations).

In *CondExprs*, the first expression must have type `Success` or `Bool`. In the latter case, Boolean expressions in conditions are considered as an abbreviation for a (nested) `if-then-else` (see Section 2.3.2).

If the local definitions in a `let` expression contain at least one declaration of a free variable, the expression (after the keyword `in`) must be of type `Success`.

In qualified names (e.g., *QFunctionID*), no characters (e.g., spaces) are allowed between the dot and the module and entity names. On the other hand, an infix expression (*Expr QInfixOpID Expr*) must contain at least one space or similar character after the left expression if the infix operator starts with a dot.

C.4 Infix Operators

In the grammar above, the use of infix operators in the rule for *Expr* is ambiguous. These ambiguities are resolved by assigning an associativity and precedence to each operator (*InfixOpID*) by a *fixity declaration*. There are three kinds of associativities, non-, left- and right-associativity (`infix`, `infixl`, `infixr`) and ten levels of precedence, 0 to 9, where level 0 binds least tightly and level 9 binds most tightly. All fixity declarations must appear at the beginning of a module. Any operator without an explicit fixity declaration is assumed to have the declaration `infixl 9`. For instance, the expression "`1+2*3+4==x && b`" is equivalent to "`((((1+(2*3))+4)==x) && b)`" w.r.t. the fixity declarations provided in the prelude.

Note that the correct use of precedences and associativities excludes some expressions which are valid w.r.t. the context-free grammar. In general, the arguments of an infix operator must have an infix operator at the top with a higher precedence than the current infix operator (or no infix operator at the top). Additionaly, the left or right argument of a left- or right-associative operator can have a top infix operator with the same precedence. The unary minus operator is interpreted with precedence 6 which means that its argument must have a precedence of at least 7. The expression "`(-` t`)`" is not considered as a right section but as the negation of t. As a consequence, the expressions "`1<2==True`" and "`1 + - 3`" are not allowed and must be bracketed as "`(1<2)==True`" and "`1 + (- 3)`".

D Operational Semantics of Curry

The precise specification of the operational semantics of Curry is based on the evaluation annotation and the patterns on the rules' left-hand sides for each function. Therefore, we describe the computation model by providing a precise definition of pattern matching with evaluation annotations (Section D.1) which is the basis to define the computation steps on expressions (Section D.2). The extension of this basic computation model to solving equational constraints and higher-order functions is described in Sections D.4 and D.5, respectively. Section D.6 describes the automatic generation of the definitional trees to guide the pattern matching strategy. Finally, Section D.7 specifies the operational semantics of the primitive operator `try` for encapsulating search.

D.1 Definitional Trees

We start by considering only the unconditional first-order part of Curry, i.e., rules do not contain conditions and λ-abstractions and partial function applications are not allowed. We assume some familiarity with basic notions of term rewriting [14] and functional logic programming [21].

We denote by $\mathcal{C}$ the set of *constructors* (with elements c, d), by $\mathcal{F}$ the set of *defined functions* or *operations* (with elements f, g), and by $\mathcal{X}$ the set of *variables* (with elements x, y), where $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{X}$ are disjoint. An *expression* (*data term*) is a variable $x \in \mathcal{X}$ or an application $\varphi(e_1, \ldots, e_n)$ where $\varphi \in \mathcal{C} \cup \mathcal{F}$ ($\varphi \in \mathcal{C}$) has arity n and $e_1, \ldots, e_n$ are expressions (data terms).[17] The set of all expressions and data terms are denoted by $\mathcal{T}(\mathcal{C} \cup \mathcal{F}, \mathcal{X})$ and $\mathcal{T}(\mathcal{C}, \mathcal{X})$, respectively. A *call pattern* is an expression of the form $f(t_1, \ldots, t_n)$ where each variable occurs only once, $f \in \mathcal{F}$ is an n-ary function, and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{X})$. $root(e)$ denotes the symbol at the *root* of the expression e. A *position* p in an expression e is represented by a sequence of natural numbers, $e|_p$ denotes the *subterm* or *subexpression* of e at position p, and $e[e']_p$ denotes the result of *replacing the subterm* $e|_p$ by the expression e' (see [14] for details).

A *substitution* is a mapping $\sigma\colon \mathcal{X} \to \mathcal{T}(\mathcal{C} \cup \mathcal{F}, \mathcal{X})$ with $\sigma(x) \neq x$ only for finitely many variables x. Thus, we denote a substitution by $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where $\sigma(x_i) \neq x_i$ for $i = 1, \ldots, n$, and id denotes the identity substitution ($n = 0$). $\mathcal{D}om(\sigma) = \{x_1, \ldots, x_n\}$ is the *domain* of σ and

$$\mathcal{VR}an(\sigma) = \{x \mid x \text{ is a variable occurring in some } t_i, i \in \{1, \ldots, n\}\}$$

is its *variable range*. Substitutions are extended to morphisms on expressions by $\sigma(f(e_1, \ldots, e_n)) = f(\sigma(e_1), \ldots, \sigma(e_n))$ for every expression $f(e_1, \ldots, e_n)$.

A Curry program is a set of rules $l = r$ satisfying some restrictions (see Section 2.3). In particular, the left-hand side l must be a call pattern. A rewrite rule is called a *variant* of another rule if it is obtained by a unique replacement of variables by other variables.

An *answer expression* is a pair $\sigma \,[]\, e$ consisting of a substitution σ (the answer computed so far) and an expression e. An answer expression $\sigma \,[]\, e$ is *solved* if e is a data term. We sometimes omit the identity substitution in answer expressions, i.e., we write e instead of $id \,[]\, e$ if it is clear from the context. A *disjunctive expression* is a (multi-)set of answer expressions $\{\sigma_1 \,[]\, e_1, \ldots, \sigma_n \,[]\, e_n\}$. The set of all disjunctive expressions is denoted by $\mathcal{D}$.

[17] Since we do not consider partial applications in this part, we write the full application as $\varphi(e_1, \ldots, e_n)$ which is equivalent to Curry's notation $\varphi\ e_1 \ldots e_n$.

A single *computation step* performs a reduction in exactly one unsolved expression of a disjunction. To provide a precise definition of the operational semantics, we use definitional trees.[18] A definitional tree is a hierarchical structure containing all rules of a defined function. $\mathcal{T}$ is a *definitional tree with call pattern* π iff the depth of $\mathcal{T}$ is finite and one of the following cases holds:

$\mathcal{T} = rule(l = r)$, where $l = r$ is a variant of a rule such that $l = \pi$.

$\mathcal{T} = branch(\pi, o, r, \mathcal{T}_1, \ldots, \mathcal{T}_k)$, where o is an occurrence of a variable in π, $r \in \{rigid, flex\}$, $c_1, \ldots, c_k$ are different constructors of the sort of $\pi|_o$, for some $k > 0$, and, for all $i = 1, \ldots, k$, $\mathcal{T}_i$ is a definitional tree with call pattern $\pi[c_i(x_1, \ldots, x_n)]_o$, where n is the arity of c_i and $x_1, \ldots, x_n$ are new variables.

$\mathcal{T} = or(\mathcal{T}_1, \mathcal{T}_2)$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are definitional trees with call pattern π.[19]

A *definitional tree of an n-ary function* f is a definitional tree $\mathcal{T}$ with call pattern $f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are distinct variables, such that for each rule $l = r$ with $l = f(t_1, \ldots, t_n)$ there is a node $rule(l' = r')$ in $\mathcal{T}$ with l variant of l'. In the following, we write $pat(\mathcal{T})$ for the call pattern of a definitional tree $\mathcal{T}$.

It is always possible to construct a definitional tree for each function (concrete algorithms are described in [3, 33] and in Section D.6). For instance, consider the following definition of the less-or-equal predicate on natural numbers represented by data terms built from `z` (zero) and `s` (successor):

```
data Nat = z | s Nat

leq :: Nat -> Nat -> Bool
leq z     n     = True
leq (s m) z     = False
leq (s m) (s n) = leq m n
```

Consider a function call like (`leq` e_1 e_2). In order to apply some reduction rule, the first argument e_1 must always be evaluated to *head normal form* (i.e., to an expression without a defined function symbol at the top). However, the second argument must be evaluated only if the first argument has the form (`s` e).[20] This dependency between the first and the second argument is expressed by the definitional tree

$$branch(\texttt{leq}(x_1, x_2), 1, flex, rule(\texttt{leq}(\texttt{z}, x_2) = \texttt{True}),$$
$$branch(\texttt{leq}(\texttt{s}(x), x_2), 2, flex, rule(\texttt{leq}(\texttt{s}(x), \texttt{z}) = \texttt{False}),$$
$$rule(\texttt{leq}(\texttt{s}(x), \texttt{s}(y)) = \texttt{leq}(x, y))))$$

This definitional tree specifies that the first argument is initially evaluated and the second argument is only evaluated if the first argument has the constructor `s` at the top. The precise operational meaning induced by definitional trees is described in the following section.

[18] Our notion is influenced by Antoy's work [3], but here we use an extended form of definitional trees.

[19] For the sake of simplicity, we consider only binary *or* nodes. The extension to such nodes with more than two subtrees is straightforward.

[20] Naive lazy narrowing strategies may also evaluate the second argument in any case. However, as shown in [6], the consideration of dependencies between arguments is essential to obtain optimal evaluation strategies.

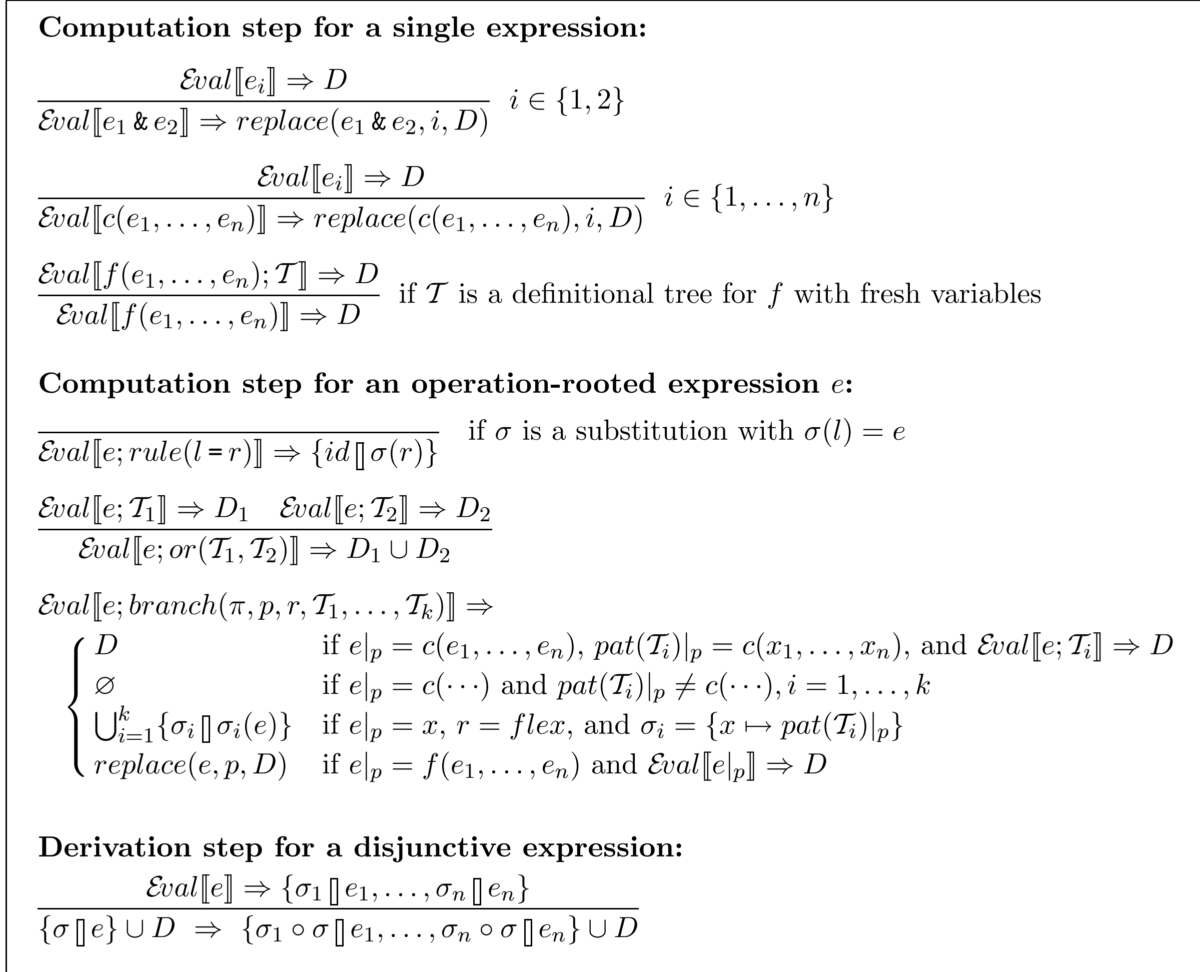


Figure 2: Operational semantics of Curry

D.2 Computation Steps

The operational semantics is defined by the derivability relation $D_1 \Rightarrow D_2$ on disjunctive expressions specified in Figure 2. An inference rule $\frac{\alpha}{\beta}$ should be read as "if α is derivable, then β is also derivable". We say that the computation of an expression e *suspends* if there is no D with $\mathcal{E}val[\![e]\!] \Rightarrow D$. A constraint is solvable if it can be reduced to `success`. The exact method to solve constraints depends on the constraint solver. A method to solve equational constraints on data terms is specified in Section D.4. For the purpose of this definition, we consider `success` as the neutral element of the operation `&`, i.e., `success` $\&\, c$ and $c\, \&$ `success` are considered as equivalent to c.

As can be seen by the rules for evaluating constraints in Figure 2, the concurrent conjunction of two constraints is evaluated by applying computation steps to one of the two constraints. Since both constraints must be finally solved (i.e., reduced to `success`), one can consider the evaluation of a concurrent conjunction as two computation threads evaluating these constraints. They are synchronized by accessing common variables (which may suspend a thread, see below). In a simple sequential implementation, the evaluation of $e_1 \,\&\, e_2$ could be started by an attempt to solve e_1. If the evaluation of e_1 suspends, an evaluation step is applied to e_2. If a variable responsible

to the suspension of e_1 is instantiated, the left expression e_1 will be evaluated in the subsequent step. Thus, we obtain a concurrent behavior with an interleaving semantics. However, a sophisticated implementation should provide a fair selection of threads, e.g., as done in a Java-based implementation of Curry [25].

A computation step for an expression e attempts to apply a reduction step to an outermost operation-rooted subterm in e by evaluating this subterm with the definitional tree for the function symbol at the root of this subterm. Although it is unspecified which outermost subterm is evaluated (compare second inference rule in Figure 2 for constructor-rooted expressions), we assume that a single (e.g., leftmost) outermost subterm is always selected. The evaluation of an operation-rooted term is defined by a case distinction on the definitional tree. If it is a *rule* node, this rule is applied. An *or* node produces a disjunction where the different alternatives are combined. Note that this definition requires that the entire disjunction suspends if one disjunct suspends. This can be relaxed in concrete implementations by continuing the evaluation in one *or* branch even if the other branch is suspended. However, to ensure completeness, it is not allowed to omit a suspended *or* branch and continue with the other non-suspended *or* branch [23]. For a similar reason, we cannot commit to a disjunct which does not bind variables but we have to consider both alternatives (see [5] for a counter-example).

The most interesting case is a *branch* node. Here we have to branch on the value of the top-level symbol at the selected position. If the symbol is a constructor, we proceed with the appropriate definitional subtree, if possible. If it is a function symbol, we proceed by evaluating this subexpression. If it is a variable and the branch is flexible, we instantiate the variable to the different constructors, otherwise (if the branch is *rigid*) we cannot proceed and suspend. The auxiliary function *replace* puts a possibly disjunctive expression into a subterm:

$$replace(e, p, \{\sigma_1 \,[]\, e_1, \ldots, \sigma_n \,[]\, e_n\}) = \{\sigma_1 \,[]\, \sigma_1(e)[e_1]_p, \ldots, \sigma_n \,[]\, \sigma_n(e)[e_n]_p\} \qquad (n \geq 0)$$

The overall computation strategy on disjunctive expressions takes a disjunct $\sigma \,[]\, e$ not in solved form and evaluates e. If the evaluation does not suspend and yields a disjunctive expression, we substitute it for $\sigma \,[]\, e$ composed with the old answer substitution.

Soundness and completeness results for the operational semantics can be found in [23]. Note that the evaluation of expressions is completely deterministic if the concurrent conjunction of constraints does not occur.

Conditional rules. Note that the operational semantics specified in Figure 2 handles only unconditional rules. Conditional rules are treated by the rules in Figure 3. If a conditional rules is applied, the condition and the right-hand side are joined into a *guarded expression*, i.e., a pair of the form $c|r$. Due to the additional evaluation rules, a guarded expression is evaluated by an attempt to solve its condition. If the condition is solvable (i.e., reducible to `success`), the guarded expression is replaced by its right-hand side. If the condition is not solvable and does not suspend, the disjunct containing this guarded expression will be deleted by the definition of computation steps, i.e., the application of the conditional rule fails and does not contribute to the final result.

Sharing and graph rewriting. For the sake of simplicity, this description of the operational semantics is based on term rewriting and does not take into account that common subterms are shared (see Section 2.3.1). We only note here that *several occurrences of the same variable are*

$$\frac{}{\mathcal{E}val[\![e; rule(l \mid c = r)]\!] \Rightarrow \{id\,[\!]\,\sigma(c \mid r)\}} \quad \text{if } \sigma \text{ is a substitution with } \sigma(l) = e$$

$$\frac{\mathcal{E}val[\![c]\!] \Rightarrow D}{\mathcal{E}val[\![c \mid e]\!] \Rightarrow replace(c \mid e, 1, D)} \qquad \frac{}{\mathcal{E}val[\![\texttt{success} \mid e]\!] \Rightarrow \{id\,[\!]\,e\}}$$

Figure 3: Evaluation of conditional rules

always shared, i.e., if an argument of a function is instantiated during a call to this function to an expression and this expression is evaluated to some value (head normal form), then all other expressions resulting from instantiating occurrences of the same argument are replaced by the same value (head normal form).[21] This is necessary not only for efficiency reasons but also for the soundness of the operational semantics in the presence of non-deterministic functions, as discussed in Section 2.3.1. The sharing of variables can be described with the more complicated framework of graph rewriting. Formal descriptions of graph rewriting and narrowing can be found in [15, 18].

D.3 Committed Choice

To define the semantics of calls to functions with the evaluation annotation `choice`, we assume that all such calls are enclosed by $choice(\ldots)$. Now we extend the rules in Figures 2 and 3 for covering such choices. An expression $choice(f(e_1, \ldots, e_n))$ is reduced as follows. If

$$\mathcal{E}val[\![f(e_1, \ldots, e_n)]\!] \;\Rightarrow^*\; \{\ldots, \sigma\,[\!]\,\texttt{success} \mid \varphi(r), \ldots\}$$

according to the operational semantics described in Figures 2 and 3, where $l \mid c = r$ is a rule for f,[22] σ does not bind any free variable occurring in $f(e_1, \ldots, e_n)$, and this is the first step in the derivation with this property, then $\mathcal{E}val[\![choice(f(e_1, \ldots, e_n))]\!] \Rightarrow \{id\,[\!]\,\varphi(r)\}$.

Thus, if disjunctions occur due to the evaluation of a call to a choice function, these disjunctions are not distributed to the top-level but are kept inside the *choice* expression. If one rule for the choice function becomes applicable (without binding of free variables in this call), all other alternatives in the disjunction are discarded. Hence, a call to a choice function corresponds to the committed choice (with deep guards) of concurrent logic languages. This can be implemented by evaluating this call (with a fair strategy for alternatives) as usual but with the restriction that only local variables can be bound.

D.4 Equational Constraints

An equational constraint e_1`=:=`e_2 is solvable if both sides are reducible to a unifiable data term (*strict equality*). An equational constraint can be solved in an incremental way by an interleaved

[21] It should be noted that values are constructor terms like `23`, `True`, or `[2,4,5]`. This means that the evaluation of constraints and I/O actions are not shared since they are not replaced by a value after evaluation but constraints are solved in order to apply a conditional rule (in case of constraints) and I/O actions are applied to the outside world when they appear at the top-level in a program. This is the intended behavior since the expressions "`putChar 'a' >> putChar 'a'`" and "`let ca = putChar 'a' in ca >> ca`" should have an identical behvarior, namely printing the character `'a'` twice.

[22] An unconditional rule $l = r$ is considered here as an abbreviation for $l \mid$ `success` $= r$.

$$\frac{\mathcal{E}val[\![e_i]\!] \Rightarrow D}{\mathcal{E}val[\![e_1\texttt{=:=}e_2]\!] \Rightarrow replace(e_1\texttt{=:=}e_2, i, D)} \quad \text{if } e_i = f(t_1,\ldots,t_n),\ i \in \{1,2\}$$

$$\overline{\mathcal{E}val[\![c(e_1,\ldots,e_n)\texttt{=:=}c(e'_1,\ldots,e'_n)]\!] \Rightarrow \{id \,[\!]\, e_1\texttt{=:=}e'_1 \,\texttt{\&}\ldots\texttt{\&}\, e_n\texttt{=:=}e'_n\}}$$

$$\overline{\mathcal{E}val[\![c(e_1,\ldots,e_n)\texttt{=:=}d(e'_1,\ldots,e'_m)]\!] \Rightarrow \varnothing} \quad \text{if } c \neq d \text{ or } n \neq m$$

$$\frac{\mathcal{E}val[\![x\texttt{=:=}e]\!] \Rightarrow D}{\mathcal{E}val[\![e\texttt{=:=}x]\!] \Rightarrow D} \quad \text{if } e \text{ is not a variable}$$

$$\overline{\mathcal{E}val[\![x\texttt{=:=}y]\!] \Rightarrow \{\{x \mapsto y\} \,[\!]\, \texttt{success}\}}$$

$$\overline{\mathcal{E}val[\![x\texttt{=:=}c(e_1,\ldots,e_n)]\!] \Rightarrow \{\sigma \,[\!]\, y_1\texttt{=:=}\sigma(e_1) \,\texttt{\&}\ldots\texttt{\&}\, y_n\texttt{=:=}\sigma(e_n)\}} \quad \begin{array}{l}\text{if } x \notin cv(c(e_1,\ldots,e_n)),\\ \sigma = \{x \mapsto c(y_1,\ldots,y_n)\},\\ y_1,\ldots,y_n \text{ fresh variables}\end{array}$$

$$\overline{\mathcal{E}val[\![x\texttt{=:=}c(e_1,\ldots,e_n)]\!] \Rightarrow \varnothing} \quad \text{if } x \in cv(c(e_1,\ldots,e_n))$$

Figure 4: Solving equational constraints

lazy evaluation of the expressions and binding of variables to constructor terms. The evaluation steps implementing this method are shown in Figure 4, where we consider the symbol `=:=` is a binary infix function symbol. The last two rules implements an occur check where the *critical variables* (cv) are only those variables occurring outside a function call:

$$\begin{aligned} cv(x) &= \{x\} \\ cv(c(e_1,\ldots,e_n)) &= cv(e_1) \cup \ldots \cup cv(e_n) \\ cv(f(e_1,\ldots,e_n)) &= \varnothing \end{aligned}$$

However, Curry implementations can also provide other methods to solve equational constraints or other types of constraints with appropriate constraint solvers. The only property, which is important for the operational semantics, is the fact that a solved constraint has the form `success`, i.e., a solvable constraint c is reducible to the answer expression $\sigma \,[\!]\, \texttt{success}$ where σ is a solution of c.

D.5 Higher-Order Features

Warren [49] has shown for the case of logic programming that the higher-order features of functional programming can be implemented by providing a (first-order) definition of the application function. Since Curry supports the higher-order features of current functional languages but excludes the guessing of higher-order objects by higher-order unification (as, e.g., in [24, 38, 43]), the operational semantics specified in Figure 2 can be simply extended to cover Curry's higher-order features by adding the following axiom (here we denote by the infix operator "`@`" the application of a function

to an expression):

$$\mathcal{E}val[\![\varphi(e_1,\ldots,e_m)\texttt{@}e]\!] \Rightarrow \varphi(e_1,\ldots,e_m,e) \quad \text{if } \varphi \text{ has arity } n \text{ and } m<n$$

Thus, we evaluate an application by simply adding the argument to a partial application. If a function has the right number of arguments, it is evaluated by the rules in Figure 2. Note that the function application is "rigid" in its first argument since the application suspends if the applied function is unknown (instead of a possible but expensive non-deterministic search for the appropriate function).

λ-abstractions occurring in expressions are anonymous functions and can be implemented by assigning a new name to each λ-abstraction, where the free variables become parameters of the function. For instance, the λ-abstraction (`f` is a free variable)

```
\x y -> f x y + 2
```

can be replaced by the (partial) application `(lambda f)`, where `lambda` is a new name, and adding the rule

```
lambda f x y = f x y + 2
```

Note that this transformation (as well as the extension of $\mathcal{E}val[\![\cdot]\!]$ above) is only used to explain the operational meaning of Curry's higher-order features. A concrete Curry system is free to choose other implementation techniques.

D.6 Generating Definitional Trees

Curry's computation model is based on the specification of a definitional tree for each operation. Although definitional trees are a high-level formalism to specify evaluation strategies, it is tedious to annotate each operation which its definitional tree. Therefore, the user need not write them explicitly in the program since they are automatically inserted by the Curry system. In the following, we present Curry's algorithm to generate definitional trees from the left-hand sides of the functions' rules.

The generation of definitional trees for each function is not straightforward, since there may exist many non-isomorphic definitional trees for a single function representing different evaluation strategies. This demands for a default strategy to generate definitional trees. Curry uses the following default strategy:

1. The leftmost argument, where a constructor occurs at the corresponding position in all left-hand sides defining this function, is evaluated to head normal form.

2. *or* nodes (i.e., disjunctions) are generated in case of a conflict between constructors and variables in the left-hand sides, i.e., if two rules have a variable and a constructor at the same position on the left-hand side.

In the following, we assume that all rules are unconditional (it is obvious how to extend it to conditional rules since only the left-hand sides of the rules are relevant for the definitional trees). To specify the construction algorithm, we define by

$$DP(\pi,R) = \{o \text{ position of a variable in } \pi \mid root(l|_o) \in \mathcal{C} \text{ for some } l \,\texttt{=}\, r \in R\}$$

the set of *demanded positions* of a call pattern π w.r.t. a set of rules R. For instance, the demanded positions of the call pattern `leq(x,y)` w.r.t. the rules for the predicate `leq` (see Section 17, page 65) are $\{1, 2\}$ referring to the pattern variables `x` and `y`. Furthermore, we define by

$$IP(\pi, R) = \{o \in DP(\pi, R) \mid root(l|_o) \in \mathcal{C} \text{ for all } l = r \in R\}$$

the set of *inductive positions* of a call pattern π w.r.t. a set of rules R. Thus, the inductive positions are those demanded positions where a constructor occurs in all left-hand sides defining this function. For instance, the set of inductive positions of the call pattern `leq(x,y)` w.r.t. the rules for the predicate `leq` is $\{1\}$.

The *g*eneration of a definitional *t*ree for a call pattern π and a set of rules R (where l is an instance of π for each $l = r \in R$) is described by the function $gt(\pi, m, R)$ ($m \in \{flex, rigid\}$ determines the mode annotation in the generated *branch* nodes). We distinguish the following cases for gt:

1. If the position o is leftmost in $IP(\pi, R)$, $\{root(l|_o) \mid l = r \in R\} = \{c_1, \ldots, c_k\}$ where $c_1, \ldots, c_k$ are different constructors with arities $n_1, \ldots, n_k$, and $R_i = \{l = r \in R \mid root(l|_o) = c_i\}$, then

$$\begin{aligned} gt(\pi, m, R) = branch(\pi, o, m, & \; gt(\pi[c_1(x_{11}, \ldots, x_{1n_1})]_o, m, R_1), \\ & \ldots, \\ & \; gt(\pi[c_k(x_{k1}, \ldots, x_{kn_k})]_o, m, R_k)) \end{aligned}$$

   where x_{ij} are fresh variables. I.e., we generate a *branch* node for the leftmost inductive position (provided that there exists such a position).

2. If $IP(\pi, R) = \varnothing$, let o be the leftmost position in $DP(\pi, R)$ and $R' = \{l = r \in R \mid root(l|_o) \in \mathcal{C}\}$. Then

$$gt(\pi, m, R) = or(gt(\pi, m, R'), gt(\pi, m, R - R'))$$

   I.e., we generate an *or* node if the leftmost demanded position of the call pattern is not demanded by the left-hand sides of all rules.

3. If $DP(\pi, R) = \varnothing$ and $l = r$ variant of some rule in R with $l = \pi$, then

$$gt(\pi, m, R) = rule(l = r)$$

   Note that all rules in R are variants of each other if there is no demanded position (this follows from the weak orthogonality of the rewrite system). For non-weakly orthogonal rewrite systems, which may occur in the presence of non-deterministic functions [18] or conditional rules, the rules in R may not be variants. In this case we simply connect the different rules by *or* nodes.

If R is the set of all rules defining the n-ary function f, then a definitional tree for f is generated by computing $gt(f(x_1, \ldots, x_n), m, R)$, where $m = rigid$ if the type of f has the form $\cdots$`->IO ...`, i.e., if f denotes an IO action, and $m = flex$ otherwise (this default can be changed by the evaluation annotation "`eval rigid`", see Section 3).

$$\mathcal{E}val[\![\texttt{try}(g)]\!] \Rightarrow \begin{cases} \texttt{[]} & \text{if } \mathcal{E}val[\![c]\!] \Rightarrow \varnothing \\ \texttt{[}g'\texttt{]} & \text{if } \mathcal{E}val[\![c]\!] \Rightarrow \{\sigma \,[\!]\, \texttt{success}\} \text{ (i.e., } \sigma \text{ is a mgu for all equations in } c\text{) with} \\ & \mathcal{D}om(\sigma) \subseteq \{x, x_1, \ldots, x_n\} \text{ (or } c = \texttt{success} \text{ and } \sigma = id\text{) and} \\ & g' = \texttt{\textbackslash}x\texttt{->let } x_1, \ldots, x_n \texttt{ free in } \sigma \circ \varphi \,[\!]\, \texttt{success} \\ D & \text{if } \mathcal{E}val[\![c]\!] \Rightarrow \{\sigma \,[\!]\, c'\} \text{ with } \mathcal{D}om(\sigma) \subseteq \{x, x_1, \ldots, x_n\}, \\ & g' = \texttt{\textbackslash}x\texttt{->let } x_1, \ldots, x_n, y_1, \ldots, y_m \texttt{ free in } \sigma \circ \varphi \,[\!]\, c' \\ & \text{where } \{y_1, \ldots, y_m\} = \mathcal{VR}an(\sigma) \setminus (\{x, x_1, \ldots, x_n\} \cup free(g)), \\ & \text{and } \mathcal{E}val[\![\texttt{try}(g')]\!] \Rightarrow D \\ \texttt{[}g_1\texttt{,}\ldots\texttt{,}g_k\texttt{]} & \text{if } \mathcal{E}val[\![c]\!] \Rightarrow \{\sigma_1 \,[\!]\, c_1, \ldots, \sigma_k \,[\!]\, c_k\},\ k > 1, \text{ and, for } i = 1, \ldots, k, \\ & \mathcal{D}om(\sigma_i) \subseteq \{x, x_1, \ldots, x_n\} \text{ and} \\ & g_i = \texttt{\textbackslash}x\texttt{->let } x_1, \ldots, x_n, y_1, \ldots, y_{m_i} \texttt{ free in } \sigma_i \circ \varphi \,[\!]\, c_i \\ & \text{where } \{y_1, \ldots, y_{m_i}\} = \mathcal{VR}an(\sigma_i) \setminus (\{x, x_1, \ldots, x_n\} \cup free(g)) \end{cases}$$

Figure 5: Operational semantics of the `try` operator for $g = \texttt{\textbackslash}x\texttt{->let } x_1, \ldots, x_n \texttt{ free in } \varphi \,[\!]\, c$

It is easy to see that this algorithm computes a definitional tree for each function since the number of rules is reduced in each recursive call and it keeps the invariant that the left-hand sides of the current set of rules are always instances of the current call pattern.

Note that the algorithm *gt* is not strictly conform with the pattern matching strategy of functional languages like Haskell or Miranda, since it generates for the rules

```
f 0 0 = 0
f x 1 = 0
```

the definitional tree

$$branch(\texttt{f(x1,x2)}, 2, flex, branch(\texttt{f(x1,0)}, 1, flex, rule(\texttt{f(0,0) = 0})), \\ rule(\texttt{f(x1,1) = 0}))$$

(although both arguments are demanded, only the second argument is at an inductive position) whereas Haskell or Miranda have a strict left-to-right pattern matching strategy which could be expressed by the definitionl tree

$$or(branch(\texttt{f(x1,x2)}, 1, flex, branch(\texttt{f(0,x2)}, 2, flex, rule(\texttt{f(0,0) = 0}))), \\ branch(\texttt{f(x1,x2)}, 2, flex, rule(\texttt{f(x1,1) = 0})))$$

The latter tree is not optimal since it has a non-deterministic *or* node and always requires the evaluation of the first argument (in the first alternative). If the function definitions are *uniform* in the sense of [48], the strategy described by *gt* is identical to traditional functional languages.

D.7 Encapsulated Search

The exact behavior of the `try` operator is specified in Figure 5. Note that a substitution φ of the form $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, computed by evaluating the body of a search goal, must be encoded as a constraint in the new search goal. Therefore, φ is transformed into its *equational representation* $\overline{\varphi}$, where $\overline{\varphi}$ is a solved constraint of the form x_1`=:=`t_1`&...&`x_n`=:=`t_n. Hence, a search goal can

always be written as try $\backslash x$->$\overline{\varphi}$ & c. Since the solved constraint $\overline{\varphi}$ does not have to be evaluated again, we use the sugared form try $\backslash x$->$\varphi \,[\!]\, c$ where only c is further evaluated. Thus, an initial search goal try $\backslash x$->c is equivalent to try $\backslash x$->$id \,[\!]\, c$.

A search goal is solved (second case) if the constraint is solvable without bindings of global variables. In a deterministic step (third case), we apply the try operator again after adding the newly introduced variables to the list of local variables. Note that the *free variables* occurring in g, denoted by $free(g)$, must not be locally declared because they can appear also outside of g, and therefore they have to be removed from $\mathcal{VR}an(\sigma)$. In a non-deterministic step (fourth case), we return the different alternatives as the result. Note that the evaluation of the try operator suspends if a computation step on the constraint attempts to bind a free variable. In order to ensure that an encapsulated search will not be suspended due to necessary bindings of free variables, the search goal should be a closed expression when a search operator is applied to it, i.e., the search variable is bound by the lambda abstraction and all other free variables are existentially quantified by local declarations.

If one wants to use free variables in a search goal, e.g., for synchronization in systems where concurrency is intensively used, the goal should perform only deterministic steps until all free variables have been bound by another part of the computation. Otherwise, a non-deterministic step could unnecessarily split the search goal. For instance, consider the expression

```
try \x -> y=:=[x] & append x [0,1] =:= [0,1]
```

The computation of y=:=[x] suspends because y is free. Therefore, a concurrent computation of the conjunction will split the goal by reducing the append expression. However, if y is first bound to [[]] by another thread of the computation, the search goal could proceed deterministically.

To avoid such unnecessary splittings, we can restrict the applicability of the split case and adapt a solution known as *stability* from AKL [28] and Oz [45]. For this purpose, we can slightly change the definition of try such that non-deterministic steps lead to a suspension as long as a deterministic step might be enabled by another part of the computation, i.e., the search goal contains free global variables. To do so, we have to replace only the fourth rule of the try operator:

$$\begin{array}{ll} \mathcal{E}val[\![\texttt{try}(g)]\!] \Rightarrow \texttt{[}g_1\texttt{,}\ldots\texttt{,}g_k\texttt{]} & \text{if } \mathcal{E}val[\![c]\!] \Rightarrow \{\sigma_1 \,[\!]\, c_1, \ldots, \sigma_k \,[\!]\, c_k\},\ k > 1, \\ & \mathcal{E}val[\![c]\!] \not\Rightarrow \{\sigma \,[\!]\, c'\}, free(c) \subseteq \{x, x_1, \ldots, x_n\} \\ & \text{and, for } i = 1, \ldots, k, \\ & g_i = \backslash x\texttt{->let } x_1, \ldots, x_n, y_1, \ldots, y_{m_i} \texttt{ free in } \sigma_i \circ \varphi \,[\!]\, c_i \\ & \text{where } y_1, \ldots, y_{m_i} = \mathcal{VR}an(\sigma_i) \setminus (\{x, x_1, \ldots, x_n\} \cup free(g)) \end{array}$$

Thus, a non-deteterministic step is only performed if no deterministic step is possible and the unsolved constraint c contains no free variables except those locally declared in the search goal. Global variables appearing in the data terms t_i of the substitution $\varphi = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ but not in c are not considered by these conditions because they cannot influence the further evaluation of c. Note that the former condition $\mathcal{D}om(\sigma_i) \subseteq \{x, x_1, \ldots, x_n\}$ is covered by the new condition $free(c) \subseteq \{x, x_1, \ldots, x_n\}$ because $\mathcal{D}om(\sigma_i)$ is always a subset of $free(c)$.

Note that the first definition of try in Figure 5 computes non-deterministic splittings where the latter definition suspends. On the other hand, the latter definition can avoid some unnecessary splittings. Thus, different implementations of Curry can support one or both of these definitions.

D.8 Eliminating Local Declarations

In the description of the operational semantics above, we assumed that a Curry program is a set of rules $l = r$ where the left-hand side l is a call pattern and the right-hand side r is an expression containing variables, constructors, and defined functions. However, a Curry program can also contain local declarations (cf. Section 2.4) whose operational semantics is not defined so far. To simplify the operational semantics, we do not extend it to local declarations but provide in the following a transformation of Curry programs containing local declarations into Curry programs without local declarations (except for free variables). The main purpose of this transformation is to provide a precise definition of the operational semantics of the full language. This transformation can also be performed by implementations of Curry but it is also possible that some implementations provide explicit support for local declarations, provided that they satisfy the operational meaning described in the following.

The elimination of local function and pattern declarations is done by the following steps which are subsequently described in more detail:

1. Eliminate sections and lambda abstractions
2. Eliminate Boolean guards in rules
3. Transform `where` declarations into `let` declarations
4. Eliminate local patterns and functions

For the following, we assume that all name conflicts have been resolved, i.e., the names of functions and argument variables defined in the different declarations are pairwise different.

Eliminate sections and lambda abstractions:

Each *left section* `(`*expr op*`)` is transformed into the partial application `((`*op*`)` *expr*`)`.

Each *right section* `(`*op expr*`)` is transformed into the lambda abstraction `(\`x`->(`*op*`)` x *expr*`)` where x is a new variable.

All λ-abstractions are eliminated by providing a new name (cf. Section D.5), i.e., each λ-abstraction `(\`$x_1 \ldots x_n$`->`e`)` is replaced by the expression `(let` $f\ x_1 \ldots x_n$ `=` e `in` f`)` where f is a new function name. This transformation must be recursively applied to all λ-abstractions occurring in e.

Eliminate Boolean guards:

This is done according the meaning described in Section 2.3.2. A rule of the form

$f\ t_1 \ldots t_n$ `|` b_1 `=` e_1
$\vdots$
`|` b_k `=` e_k
`where` *decls*

(the `where` part can also be missing), where all guards $b_1, \ldots, b_k$ ($k > 0$) are expressions of type `Bool`, is transformed into the single rule

$f\ t_1 \ldots t_n$ `= if` b_1 `then` e_1 `else`

$\vdots$

```
if b_k then e_k else undefined
where decls
```

Transform `where` into `let`:

Each unconditional rule of the form

l `=` r `where` *decls*

is transformed into

l `=` `let` *decls* `in` r .

Each conditional rule of the form

l `|` c `=` r [`where` *decls*]

is transformed into

l `=` [`let` *decls* `in`] `(`c`|`r`)`

where the meaning of the guarded expression `(`c`|`r`)` is explained in Figure 3. Thus, we assume in the subsequent transformations that all program rules are of the form "l `=` r" (where r might be a let-expression).

Note that this transformation is not really necessary but provides for a unified treatment of the elimination of local pattern and function declarations in `let` and `where`.

Eliminate local patterns and functions:

All local pattern and function declarations in a rule "l `=` r" are eliminated by iterating the elimination of outermost local declarations as follows. For this purpose, we denote by $r = C[$`let` *decls* `in` $e]_p$ an outermost `let` declaration in r, i.e., there is no prefix position $p' < p$ in r with $r|_{p'} =$ `let`$\ldots$

We apply the following transformations as long as possible to eliminate all local pattern and function declarations in a rule of the form "l `=` $C[$`let` *decls* `in` $e]_p$":

Eliminate patterns:

Select a local pattern declaration which contains only argument variables from the main function's left-hand side in the expression, i.e., the rule has the form

```
l = C[let decls_1
          p = e'
          decls_2
      in e]_p
```

with $free(e') \subseteq free(l)$ (it is a programming error if no pattern declaration has this property, i.e., cyclic pattern definitions or pattern definitions depending on locally free variables are not allowed). Then transform this rule into the rules

l `=` $C[f'\ x_1 \ldots x_k\ e']_p$
$f'\ x_1 \ldots x_k\ z$ `=` $f''\ x_1 \ldots x_k\ (f_1\ z) \ldots (f_m\ z)$

$$f'' \;\; x_1 \ldots x_k \;\; y_1 \ldots y_m \;\; \texttt{=} \;\; \texttt{let} \;\; decls_1$$
$$decls_2$$
$$\texttt{in} \;\; e$$
$$f_1 \;\; p \;\; \texttt{=} \;\; y_1$$
$$\ldots$$
$$f_m \;\; p \;\; \texttt{=} \;\; y_m$$

where $x_1 \ldots x_k$ are all the variables occurring in l, $y_1 \ldots y_m$ are all the variables occurring in p, z is a new variable symbol, and f', f'', $f_1, \ldots, f_m$ are new function symbols. Repeat this step for f'' until all local pattern declarations are eliminated.

This translation can be optimized in some specific cases. If p is just a variable, the function f' is not needed and the definition of l can be simplified into

$$l \;\; \texttt{=} \;\; C[f'' \;\; x_1 \ldots x_k \;\; e']_p$$

Similarly, if e' is a variable, the function f' is also not needed and the definition of l can be replaced by

$$l \;\; \texttt{=} \;\; C[f'' \;\; x_1 \ldots x_k \;\; (f_1 \;\; e') \ldots (f_m \;\; e')]_p$$

Complete local function definitions:

If a locally declared function f refers in its definition to a variable v not contained in its argument patterns, then add v as an additional first[23] argument to all occurrences of f (i.e., left-hand sides and calls). Repeat this step until all locally defined functions are completed. Note that this completion step must also be applied to free variables introduced in the same `let` expression, i.e., the rule

```
f x = let y free
          g z = c y z
      in g 0
```

is completed to

```
f x = let y free
          g y z = c y z
      in g y 0
```

Globalize local function definitions:

If the definitions of all locally declared functions are completed, i.e., the definitions only refer to variables in the argument patterns, delete the locally declared functions and define them at the top-level (and rename them if there are already top-level functions with the same name). For instance, the previous rule is transformed into the definitions

```
g y z = c y z
f x = let y free in g y 0
```

Note that the entire transformation process must be applied again to the new top-level declarations since they may also contain local declarations.

[23]The new arguments must come first because of possible partial applications of this function.

After applying these transformation steps to all rules in the program, we obtain a program without sections and λ-abstractions where all local declarations contain only free variables. Note that partial applications are not eliminated since they can be treated as shown in Section D.5.

The final program might not be a valid Curry program due to the transformation of `where` declarations into `let` declarations. In order to obtain a valid Curry program, we can perform the following final transformations:

Transform	*l* `=` `let` *decls* `in` `(`*c*`|`*r*`)`	into	*l* `|` *c* `=` *r* `where` *decls*
and	*l* `=` `let` *decls* `in` *r*	into	*l* `=` *r* `where` *decls*

The latter transformation is only necessary if r is not of type `Success`.

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Index

!!, 49
$\mathcal{D}om$, 64
$\mathcal{VR}an$, 64
(), 18, 48
(,...,), 18
(...), 48
*, 54
+, 54
++, 17, 49
-, 54
--, 59
->, 16
., 27, 46
.., 20
/=, 48
:, 17, 48
<, 54
<=, 54
=, 5
=:=, 10, 54, 68
==, 14
>, 54
>=, 54
>>, 28, 55
>>=, 28, 55
[], 17, 48
[a], 17, 48
$, 16, 47
$!, 16, 47
&, 10, 15, 54
&>, 15, 55
&&, 14, 47
{-...-}, 59

abstract datatype, 24
all, 53
and, 53
annotation
    rigid, 13
answer expression, 12, 64
any, 53
appendFile, 55
application operator, 69
arithmetic sequence, 20
arity, 4

best, 32, 57
Bool, 14, 47
break, 52
browse, 33, 57
browseList, 33, 57

call pattern, 64
case, 22
case expression, 22
Char, 17
choice, 33
chr, 17, 54
comment, 59
committed choice, 33
computation step, 12
concat, 51
concatMap, 51
condition, 5
conditional, 14, 48
conditional equation, 5
conditional rule, 5
condSearch, 33, 57
confluence, 6
conjunction, 10, 14, 15, 54, 55
    concurrent, 10, 15, 54
    of constraints, 10
    sequential, 14, 15, 55
const, 46
constant, 4
constraint, 10, 15, 68
    equational, 10, 15, 68
constructor, 3, 64
curry, 46

data term, 4, 64
    ground, 4
datatype

abstract, 24
declaration, 3
declaration
datatype, 3
export, 23
fixity, 63
free, 9
function, 5
import, 24
local, 7, 8, 75
type synonym, 4
defined function, 3, 64
defining equation, 5
definitional tree, 65
disjunction, 14
disjunctive expression, 12, 64
`div`, 54
`do`, 29, 60
do notation, 29
domain, 64
`done`, 55
`doSolve`, 56
`drop`, 51
`dropWhile`, 52

`Either`, 55
`either`, 55
`elem`, 53
entity, 23
exported, 23
imported, 24
`enumFrom`, 53
`enumFromThen`, 53
`enumFromThenTo`, 54
`enumFromTo`, 54
equality
in constraints, 10
strict, 10, 14, 15, 68
test, 14
equation, 5
conditional, 5
higher-order, 6
equational constraint, 10, 15, 68
`error`, 47
`eval`, 13
evaluation annotation, 13, 33
export declaration, 23
expression, 4, 64
answer, 12, 64
case, 22
disjunctive, 12, 64
ground, 4
`external`, 35
external function, 35

`failed`, 47
`False`, 14, 47
`filter`, 50
`findall`, 32, 57
fixity declaration, 63
flexible, 13
`flip`, 47
`Float`, 17
`foldl`, 49
`foldl1`, 49
`foldr`, 49
`foldr1`, 49
$free(\cdot)$, 73
`free`, 9
free declaration, 9
free variable, 5, 9, 73
`fst`, 48
function
arity, 4
defined, 3, 64
external, 35
non-deterministic, 5
function application, 11, 69
function declaration, 5
function rule, 5

generic instance, 19
`getChar`, 27, 55
`getLine`, 27, 56
ground expression, 4
ground term, 4

`head`, 48
head normal form, 65

higher-order equation, 6

id, 46
if_then_else, 14, 48
import declaration, 24
index, 24
infix, 63
infixl, 63
infixr, 63
Int, 16
IO, 27, 55
iterate, 51

Just, 55

keywords, 59

λ-abstraction, 12, 70, 74
Left, 55
left section, 62, 74
left-hand side, 5
length, 49
let, 8, 60
lines, 52
list comprehension, 21
lists, 17, 48
local declaration, 7, 8, 75
local pattern, 8, 21, 75
lookup, 53

map, 17, 49
mapIO, 56
mapIO_, 56
Maybe, 55
maybe, 55
merge, 34
mod, 54
module, 23
module, 23
module name, 23
monadic I/O, 27

name
    of a module, 23
narrowing, 3, 13
negation, 14

non-deterministic function, 5
not, 14, 48
notElem, 53
Nothing, 55
null, 49

of, 22, 60
once, 32, 57
one, 33, 57
operation, 3, 64
or, 53
ord, 17, 54
otherwise, 7, 14, 48

pair, 48
pattern
    call, 64
    local, 8, 21, 75
position, 64
predicate, 14
prelude, 23, 46
print, 56
program, 23
putChar, 28, 55
putStr, 28, 55
putStrLn, 28, 56

range, 64
readFile, 55
repeat, 51
replicate, 51
residuation, 3, 13
return, 28, 55
reverse, 53
Right, 55
right section, 62, 74
right-hand side, 5
rigid, 13
rigid, 13, 71
root (of an expression), 64
rule, 5
    conditional, 5

search goal, 29
search variable, 29

section, 62, 74
  left, 62, 74
  right, 62, 74
`seq`, 16, 47
`sequenceIO`, 56
`sequenceIO_`, 56
sharing, 6, 68
`show`, 18, 54
`snd`, 48
`solveAll`, 32, 57
solved expression, 64
`span`, 52
`splitAt`, 51
strict equality, 10, 14, 15, 68
`String`, 18
subexpression, 64
substitution, 6, 64
subterm, 64
`Success`, 15
`success`, 15, 54
suspended evaluation, 66

`tail`, 48
`take`, 51
`takeWhile`, 51
`True`, 14, 47
`try`, 29, 57
`tryone`, 34, 57
tuple, 18, 48
type, 18
  synonym declaration, 4
type constructor, 4
type declaration, 3, 5
type environment, 19
type expression, 4, 18
type instance, 19
type scheme, 19
type synonym declaration, 4
type variable, 4

unbound variable, 9
`uncurry`, 46
unit type, 18
`unlines`, 52
`unpack`, 58
`until`, 47
`unwords`, 53
`unzip`, 50
`unzip3`, 51

variable, 64
  declaration, 9
  free, 5, 9, 73
  search, 29
variable range, 64
variant, 64

well-typed, 19
`where`, 8, 23, 60
`words`, 52
`writeFile`, 55

`zip`, 50
`zip3`, 50
`zipWith`, 50
`zipWith3`, 50